A Comprehensive Evaluation of the Differential Geometry of Cartan Connections with Metric Structure

By **Daniel L. Indranu**

Introduction

The splendid, profound, and highly intuitive interpretation of differential geometry by E. Cartan, which was first applied to Riemann spaces, has resulted in a highly systematic description of a vast range of geometric and topological properties of differentiable manifolds. Although it possesses a somewhat abstract analytical foundation, to my knowledge there is no instance where Riemann-Cartan geometry, cast in the language of differential forms (i.e., exterior calculus), gives a description that is in conflict with the classical tensor analysis as formalized, e.g., by T. Levi-Civita. Given all its successes, one might expect that any physical theory, which relies on the concept of a field, can be elegantly built on its rigorous foundation. Therefore, as long as the reality of metric structure (i.e., metric compatibility) is assumed, it appears that a substantial modified geometry is not needed to supersede Riemann-Cartan geometry.

A common overriding theme in both mathematics and theoretical physics is that of unification. And as long as physics can be thought of as geometry, the geometric objects within Riemann-Cartan geometry (such as curvature for gravity and torsion for intrinsic spin) certainly help us visualize and conceptualize the essence of unity in physics. Because of its intrinsic unity and its breadth of numerous successful applications, it might be possible for nearly all the laws governing physical phenomena to be combined and written down in compact form via the structural equations. By the intrinsic unity of Riemann-Cartan geometry, I simply refer to its tight interlock between algebra, analysis, group representation theory, and geometry. At least in mathematics alone, this is just as close as one can get to a “final” unified description of things. I believe that the unifying power of this beautiful piece of mathematics extends further still.

I’m afraid the title I’ve given to this work has a somewhat narrow meaning, unlike the way it sounds. In writing this article, my primary goal has been to present Riemann-Cartan geometry in a somewhat simpler, more concise, and therefore more efficient form than others dealing with the same subject have done before. I have therefore had to drop whatever mathematical elements or representations that might seem somewhat highly counterintuitive at first. After all, not everyone, unless perhaps he or she is a mathematician, is familiar with abstract concepts from algebra, analysis, and topology, just to name a few. Nor is he or she expected to understand these things. But one thing remains essential, namely, one’s ability to catch at least a glimpse of the beauty of the presented subject via deep, often simple, real understanding of its basics. As a non-mathematician (or simply a “dabbler” in pure mathematics), I do think that pure mathematics as a whole has grown extraordinarily “strange”, if not complex (the weight of any complexity is really relative of course), with a myriad of seemingly separate branches, each of which might only be understood at a certain level of depth by the pure mathematicians specializing in that particular branch themselves. As such, a comparable complexity may also have occurred in the case of theoretical physics itself as it necessarily feeds on the latest formalism of the relevant mathematics each time. Whatever may be the case, the real catch is in the *essential understanding* of the basics. I believe simplicity alone will reveal it without necessarily having to diminish one's perspectives at the same time. After all, this little work is intended for beginners.

1. A brief elementary introduction to the Cartan(-Hausdorff) manifold C^∞

Let $\omega_a = \frac{\partial X^i}{\partial x^a} E_i = \partial_a X^i E_i$ (summation convention employed throughout this article) be the covariant (*frame*) basis spanning the $n-$dimensional base manifold C^∞ with local coordinates $x^a = x^a\left(X^k\right)$. The contravariant (*coframe*) basis θ^b is then given via the orthogonal projection $\left\langle \theta^b, \omega_a \right\rangle = \delta^b_a$, where δ^b_a are the components of the Kronecker delta (whose value is unity if the indices coincide or null otherwise). Now the set of linearly independent local directional derivatives $E_i = \frac{\partial}{\partial X^i} = \partial_i$ gives the coordinate basis of the locally flat tangent space $T_x(M)$ at a point $x \in C^\infty$. Here M denotes the topological space of the so-called $n-$tuples $h(x) = h\left(x^1,...,x^n\right)$ such that relative to a given chart $(U, h(x))$ on a neighborhood U of a local coordinate point x, our $C^\infty-$differentiable manifold itself is a topological space. The dual basis to E_i spanning the locally flat cotangent space $T_x^*(M)$ will then be given by the differential elements dX^k via the relation $\left\langle dX^k, \partial_i \right\rangle = \delta^k_i$. In fact and in general, the *one-forms* dX^k indeed act as a linear map $T_x(M) \to IR$ when applied to an arbitrary vector field $F \in T_x(M)$ of the explicit form $F = F^i \frac{\partial}{\partial X^i} = f^a \frac{\partial}{\partial x^a}$. Then it is easy to see that $F^i = F\, X^i$ and $f^a = F\, x^a$, from which we obtain the usual transformation laws for the contravariant components of a vector field, i.e., $F^i = \partial_a X^i f^a$ and $f^i = \partial_i x^a F^i$, relating the localized components of F to the general ones and vice versa. In addition, we also see that $\left\langle dX^k, F \right\rangle = F\, X^k = F^k$.

The components of the metric tensor $g = g_{ab}\, \theta^a \otimes \theta^b$ of the base manifold C^∞ are readily given by

$$g_{ab} = \left\langle \omega_a, \omega_b \right\rangle$$

The components of the metric tensor $g\left(x_N\right) = \eta_{ik}\, dX^i \otimes dX^k$ describing the locally flat tangent space $T_x(M)$ of rigid frames at a point $x_N = x_N\left(x^a\right)$ are given by

$$\eta_{ik} = \langle E_i , E_k \rangle = diag\,(\pm 1, \pm 1, ..., \pm 1)$$

In four dimensions, the above may be taken to be the components of the Minkowski metric tensor, i.e., $\eta_{ik} = \langle E_i , E_k \rangle = diag\,(1, -1, -1, -1)$.

Then we have the expression

$$g_{ab} = \eta_{ik}\ \partial_a X^i\ \partial_b X^k$$

satisfying

$$g_{ac}\ g^{bc} = \delta_a^b$$

where $g^{ab} = \langle \theta^a , \theta^b \rangle$.

The manifold C^∞ is a metric space whose line-element in this formalism of a differentiable manifold is directly given by the metric tensor itself, i.e.,

$$ds^2 = g = g_{ab}\ \left(\partial_i x^a\ \partial_k x^b\right) dX^i \otimes dX^k$$

where the coframe basis is given by the one-forms $\theta^a = \partial_i x^a\ dX^i$.

2. Exterior calculus in n dimensions

As we know, an arbitrary tensor field $T \in C^\infty$ of rank (p, q) is the object

$$T = T^{i_1 i_2 \ldots i_q}_{j_1 j_2 \ldots j_p}\ \omega_{i_1} \otimes \omega_{i_2} \otimes \ldots \otimes \omega_{i_q} \otimes \theta^{j_1} \otimes \theta^{j_2} \otimes \ldots \otimes \theta^{j_p}$$

Given the existence of a local coordinate transformation via $x^i = x^i\left(\bar{x}^\alpha\right)$ in C^∞, the components of $T \in C^\infty$ transform according to

$$T^{ij \ldots s}_{kl \ldots r} = T^{\alpha\beta \ldots \lambda}_{\mu\nu \ldots \eta}\ \partial_\alpha\ x^i\ \partial_\beta\ x^j \ldots \partial_\lambda\ x^s\ \partial_k\ \bar{x}^\mu\ \partial_l\ \bar{x}^\nu\ \ldots\ \partial_r\ \bar{x}^\eta$$

Taking a local coordinate basis $\theta^i = dx^i$, a Pfaffian $p-$form ω is the completely anti-symmetric tensor field

$$\omega = \omega_{i_1 i_2 \ldots i_p}\ dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p}$$

where

$$dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_p} \equiv \frac{1}{p!} \delta^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_p} \; dx^{j_1} \otimes dx^{j_2} \otimes ... \otimes dx^{j_p}$$

In the above, the $\delta^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_p}$ are the components of the generalized Kronecker delta. They are given by

$$\delta^{i_1 i_2 \ldots i_p}_{j_1 j_2 \cdots j_p} = \in_{j_1 j_2 \cdots j_p} \; \in^{i_1 \ldots i_p} = \det \begin{pmatrix} \delta^{i_1}_{j_1} & \delta^{i_2}_{j_1} & \ldots & \delta^{i_p}_{j_1} \\ \delta^{i_1}_{j_2} & \delta^{i_2}_{j_2} & \ldots & \delta^{i_p}_{j_2} \\ \ldots & \ldots & \ldots & \ldots \\ \delta^{i_1}_{j_p} & \delta^{i_2}_{j_p} & \ldots & \delta^{i_p}_{j_p} \end{pmatrix}$$

where $\in_{j_1 j_2 \cdots j_p} = \sqrt{\det(g)} \; \varepsilon_{j_1 j_2 \cdots j_p}$ and $\in^{i_1 i_2 \ldots i_p} = \frac{1}{\sqrt{\det(g)}} \; \varepsilon^{i_1 i_2 \ldots i_p}$ are the covariant and contravariant components of the completely anti-symmetric Levi-Civita permutation tensor, respectively, with the ordinary permutation symbols being given as usual by $\varepsilon_{j_1 j_2 \cdots j_q}$ and $\varepsilon^{i_1 i_2 \ldots i_p}$.

We can now write

$$\omega = \frac{1}{p!} \delta^{i_1 i_2 \ldots i_p}_{j_1 j_2 \cdots j_p} \; \omega_{i_1 i_2 \ldots i_p} \; dx^{j_1} \wedge dx^{j_2} \wedge ... \wedge dx^{j_p}$$

such that for a null $p-$form $\omega = 0$ its components satisfy the relation $\delta^{i_1 i_2 \ldots i_p}_{j_1 j_2 \cdots j_p} \; \omega_{i_1 i_2 \ldots i_p} = 0$.

By meticulously moving the dx^i from one position to another, we see that

$$\begin{aligned} & dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_{p-1}} \wedge dx^{i_p} \wedge dx^{j_1} \wedge dx^{j_2} \wedge ... \wedge dx^{j_q} \\ & \qquad = (-1)^p \; dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_{p-1}} \wedge dx^{j_1} \wedge dx^{j_2} \wedge ... \wedge dx^{j_q} \wedge dx^{i_p} \end{aligned}$$

and

$$\begin{aligned} & dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_p} \wedge dx^{j_1} \wedge dx^{j_2} \wedge ... \wedge dx^{j_q} \\ & \qquad = (-1)^{pq} \; dx^{j_1} \wedge dx^{j_2} \wedge ... \wedge dx^{j_q} \wedge dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_p} \end{aligned}$$

Let ω and π be a $p-$form and a $q-$form, respectively. Then, in general, we have the following relations:

$$
\begin{aligned}
&\omega \wedge \pi = (-1)^{pq}\ \pi \wedge \omega = \omega_{i_1 i_2 \ldots i_p}\ \pi_{j_1 j_2 \cdots j_q}\ dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{p} \wedge dx^{j_1} \wedge dx^{j_2} \wedge \ldots \wedge dx^{j_q} \\
&d(\omega + \pi) = d\omega + d\pi \\
&d(\omega \wedge \pi) = d\omega \wedge \pi + (-1)^{p}\ \omega \wedge d\pi
\end{aligned}
$$

Note that the mapping $d:\omega = d\omega$ is a $(p+1)-$form. Explicitly, we have

$$
d\omega = \frac{(-1)^{p}}{(p+1)!}\ \delta^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_p}\ \frac{\partial \omega_{i_1 i_2 \ldots i_p}}{\partial x^{i_{p+1}}}\ dx^{j_1} \wedge dx^{j_2} \wedge \ldots \wedge dx^{j_p} \wedge dx^{i_{p+1}}
$$

For instance, given a (continuous) function f, the one-form $df = \partial_i f\, dx^i$ satisfies $d^2 f \equiv ddf = \partial_k \partial_i f\, dx^k \wedge dx^i = 0$. Likewise, for the one-form $A = A_i\ dx^i$, we have $dA = \partial_k\ A_i\ dx^k \wedge dx^i$ and therefore $d^2 A = \partial_l \partial_k A_i\ dx^l \wedge dx^k \wedge dx^i = 0$, i.e., $\delta^{ikl}_{rst}\ \partial_l \partial_k\ A_i = 0$ or $\partial_l \partial_k\ A_i + \partial_k \partial_i\ A_l + \partial_i \partial_l\ A_k = 0$. Obviously, the last result holds for arbitrary $p-$forms $\Pi^{ij\ldots s}_{kl\ldots r}$, i.e.,

$$
d^2 \Pi^{ij\ldots s}_{kl\ldots r} = 0
$$

Let us now consider a simple two-dimensional case. From the transformation law $dx^i = \partial_\alpha\ x^i\ d\bar{x}^\alpha$, we have, upon employing a positive definite Jacobian, i.e., $\dfrac{\partial\left(x^i, x^j\right)}{\partial\left(\bar{x}^\alpha, \bar{x}^\beta\right)} > 0$, the following:

$$
dx^i \wedge dx^j = \partial_\alpha\ x^i\ \partial_\beta\ x^j\ d\bar{x}^\alpha \wedge d\bar{x}^\beta = \frac{1}{2}\ \frac{\partial\left(x^i, x^j\right)}{\partial\left(\bar{x}^\alpha, \bar{x}^\beta\right)}\ d\bar{x}^\alpha \wedge d\bar{x}^\beta
$$

It is easy to see that

$$
dx^1 \wedge dx^2 = \frac{\partial\left(x^1, x^2\right)}{\partial\left(\bar{x}^1, \bar{x}^2\right)}\ d\bar{x}^1 \wedge d\bar{x}^2
$$

which gives the correct transformation law of a surface element.

We can now elaborate on the so-called *Stokes theorem*. Given an arbitrary function f, the integration in a domain D in the manifold C^∞ is such that

$$
\iint_D f\left(x^i\right) dx^1 \wedge dx^2 = \iint_D f\left(x^i\left(\bar{x}^\alpha\right)\right) \frac{\partial\left(x^1, x^2\right)}{\partial\left(\bar{x}^1, \bar{x}^2\right)}\ d\bar{x}^1\, d\bar{x}^2
$$

Generalizing to n dimensions, for any $\psi^i = \psi^i\left(x^k\right)$ we have

$$d\psi^1 \wedge d\psi^2 \wedge \ldots \wedge d\psi^n = \frac{\partial\left(\psi^1, \psi^2, \ldots, \psi^n\right)}{\partial\left(x^1, x^2, \ldots, \bar{x}^n\right)} dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$$

Therefore (in a particular domain)

$$\iint \ldots \int f \; d\psi^1 \wedge d\psi^2 \wedge \ldots \wedge d\psi^n = \iint \ldots \int f\left(x^i\right) \frac{\partial\left(\psi^1, \psi^2, \ldots, \psi^n\right)}{\partial\left(x^1, x^2, \ldots, x^n\right)} dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$$

Obviously, the value of this integral is independent of the choice of the coordinate system. Under the coordinate transformation given by $x^i = x^i\left(\bar{x}^\alpha\right)$, the Jacobian can be expressed as

$$\frac{\partial\left(\psi^1, \psi^2, \ldots, \psi^n\right)}{\partial\left(x^1, x^2, \ldots, x^n\right)} = \frac{\partial\left(\psi^1, \psi^2, \ldots, \psi^n\right)}{\partial\left(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n\right)} \frac{\partial\left(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n\right)}{\partial\left(x^1, x^2, \ldots, x^n\right)}$$

If we consider a $(n-m)-$ dimensional subspace (hypersurface) $S \in C^\infty$ whose local coordinates u^A parametrize the coordinates x^i, we have

$$\begin{aligned} &\iint \ldots \int f \; d\psi^1 \wedge d\psi^2 \wedge \ldots \wedge d\psi^n \\ &= \iint \ldots \int f\left(x^i\left(u^A\right)\right) \frac{\partial\left(\psi^1\left(x^i\left(u^A\right)\right), \psi^2\left(x^i\left(u^A\right)\right), \ldots, \psi^n\left(x^i\left(u^A\right)\right)\right)}{\partial\left(u^1, u^2, \ldots, u^{n-m}\right)} du^1 \, du^2 \ldots du^{n-m} \end{aligned}$$

3. Geometric properties of a curved manifold

Let us recall a few concepts from conventional tensor analysis for a while. Introducing a generally asymmetric connection Γ via the covariant derivative

$$\partial_b \, \omega_a = \Gamma^c_{ab} \; \omega_c$$

i.e.,

$$\Gamma^c_{ab} = \left\langle \theta^c, \partial_b \, \omega_a \right\rangle = \Gamma^c_{(ab)} + \Gamma^c_{[ab]}$$

where the round index brackets indicate symmetrization and the square ones indicate anti-symmetrization, we have, by means of the local coordinate transformation given by $x^a = x^a\left(\bar{x}^\alpha\right)$ in C^∞

$$\partial_b e_a^\alpha = \Gamma_{ab}^c e_c^\alpha - \overline{\Gamma}_{\beta\lambda}^\alpha e_a^\beta e_b^\lambda$$

where the tetrads of the *moving frames* are given by $e_a^\alpha = \partial_a \bar{x}^\alpha$ and $e_\alpha^a = \partial_\alpha x^a$. They satisfy $e_\alpha^a e_b^\alpha = \delta_b^a$ and $e_a^\alpha e_\beta^a = \delta_\beta^\alpha$. In addition, it can also be verified that

$$\partial_\beta e_\alpha^a = \overline{\Gamma}_{\alpha\beta}^\lambda e_\lambda^a - \Gamma_{bc}^a e_\alpha^b e_\beta^c$$
$$\partial_b e_\alpha^a = e_\lambda^a \overline{\Gamma}_{\alpha\beta}^\lambda e_b^\beta - \Gamma_{cb}^a e_\alpha^c$$

From conventional tensor analysis, we know that Γ is a non-tensorial object, since its components transform as

$$\Gamma_{ab}^c = e_\alpha^c \partial_b e_a^\alpha + e_\alpha^c \overline{\Gamma}_{\beta\lambda}^\alpha e_a^\beta e_b^\lambda$$

However, it can be described as a kind of displacement field since it is what makes possible a comparison of vectors from point to point in C^∞. In fact the relation $\partial_b \omega_a = \Gamma_{ab}^c \omega_c$ defines the so-called metricity condition, i.e., the change (during a displacement) in the basis can be measured by the basis itself. This immediately translates into

$$\nabla_c g_{ab} = 0$$

where we have just applied the notion of a covariant derivative to an arbitrary tensor field T:

$$\nabla_k T_{lm\ldots r}^{ij\ldots s} = \partial_k T_{lm\ldots r}^{ij\ldots s} + \Gamma_{pk}^i T_{lm\ldots r}^{pj\ldots s} + \Gamma_{pk}^j T_{lm\ldots r}^{ip\ldots s} + \ldots + \Gamma_{pk}^s T_{lm\ldots r}^{ij\ldots p}$$
$$- \Gamma_{lk}^p T_{pm\ldots r}^{ij\ldots s} - \Gamma_{mk}^p T_{lp\ldots r}^{ij\ldots s} - \ldots - \Gamma_{rk}^p T_{lm\ldots p}^{ij\ldots s}$$

such that $\left(\partial_k T\right)_{lm\ldots r}^{ij\ldots s} = \nabla_k T_{lm\ldots r}^{ij\ldots s}$.

The condition $\nabla_c g_{ab} = 0$ can be solved to give

$$\Gamma_{ab}^c = \frac{1}{2} g^{cd} \left(\partial_b g_{da} - \partial_d g_{ab} + \partial_a g_{bd}\right) + \Gamma_{[ab]}^c - g^{cd} \left(g_{ae} \Gamma_{[db]}^e + g_{be} \Gamma_{[da]}^e\right)$$

from which it is customary to define

$$\Delta_{ab}^c = \frac{1}{2} g^{cd} \left(\partial_b g_{da} - \partial_d g_{ab} + \partial_a g_{bd}\right)$$

as the Christoffel symbols (symmetric in their two lower indices) and

$$K^{c}_{ab} = \Gamma^{c}_{[ab]} - g^{cd} \left(g_{ae} \, \Gamma^{e}_{[db]} + g_{be} \, \Gamma^{e}_{[da]} \right)$$

as the components of the so-called contorsion tensor (anti-symmetric in the first two mixed indices).

Note that the components of the torsion tensor are given by

$$\Gamma^{a}_{[bc]} = \frac{1}{2} \, e^{a}_{\alpha} \left(\partial_c \, e^{\alpha}_{b} - \partial_b \, e^{\alpha}_{c} + e^{\beta}_{b} \, \overline{\Gamma}^{\alpha}_{\beta c} - e^{\beta}_{c} \, \overline{\Gamma}^{\alpha}_{\beta b} \right)$$

where we have set $\overline{\Gamma}^{\alpha}_{\beta c} \equiv \overline{\Gamma}^{\alpha}_{\beta \lambda} \, e^{\lambda}_{c}$.

The components of the curvature tensor R of C^{∞} are then given via the relation

$$\begin{aligned}
\left(\nabla_q \nabla_p - \nabla_p \nabla_q \right) T^{ab \ldots s}_{cd \ldots r} &= T^{ab \ldots s}_{wd \ldots r} \, R^{w}{}_{cpq} + T^{ab \ldots s}_{cw \ldots r} \, R^{w}{}_{dpq} + \ldots + T^{ab \ldots s}_{cd \ldots w} \, R^{w}{}_{rpq} \\
&\quad - T^{wb \ldots s}_{cd \ldots r} \, R^{a}{}_{wpq} - T^{aw \ldots s}_{cd \ldots r} \, R^{b}{}_{wpq} - \ldots - T^{ab \ldots w}_{cd \ldots r} \, R^{s}{}_{wpq} \\
&\quad - 2 \, \Gamma^{w}_{[pq]} \, \nabla_w \, T^{ab \ldots s}_{cd \ldots r}
\end{aligned}$$

where

$$\begin{aligned}
R^{d}{}_{abc} &= \partial_b \, \Gamma^{d}_{ac} - \partial_c \, \Gamma^{d}_{ab} + \Gamma^{e}_{ac} \, \Gamma^{d}_{eb} - \Gamma^{e}_{ab} \, \Gamma^{d}_{ec} \\
&= B^{d}{}_{abc}(\Delta) + \hat{\nabla}_b \, K^{d}_{ac} - \hat{\nabla}_c \, K^{d}_{ab} + K^{e}_{ac} \, K^{d}_{eb} - K^{e}_{ab} \, K^{d}_{ec}
\end{aligned}$$

where $\hat{\nabla}$ denotes covariant differentiation with respect to the Christoffel symbols alone, and where

$$B^{d}{}_{abc}(\Delta) = \partial_b \, \Delta^{d}_{ac} - \partial_c \, \Delta^{d}_{ab} + \Delta^{e}_{ac} \, \Delta^{d}_{eb} - \Delta^{e}_{ab} \, \Delta^{d}_{ec}$$

are the components of the Riemann-Christoffel curvature tensor of C^{∞}.

From the components of the curvature tensor, namely, $R^{d}{}_{abc}$, we have (using the metric tensor to raise and lower indices)

$$\begin{aligned}
R_{ab} &\equiv R^{c}{}_{acb} = B_{ab}(\Delta) + \hat{\nabla}_c \, K^{c}_{ab} - K^{c}_{ad} \, K^{d}_{cb} - \hat{\nabla}_b \, \Gamma^{c}_{[ac]} + K^{c}_{ab} \, \Gamma^{d}_{[cd]} + 2 \, \Gamma^{d}_{[cb]} \, \Gamma^{c}_{ad} \\
R &\equiv R^{a}{}_{a} = B(\Delta) - 2 \, g^{ab} \, \hat{\nabla}_a \, \Gamma^{c}_{[bc]} - g^{ac} \, \Gamma^{b}_{[ab]} \, \Gamma^{d}_{[cd]} - K_{abc} \, K^{acb} + 2 \, g^{ab} \, \Gamma^{d}_{[cb]} \, \Gamma^{c}_{ad}
\end{aligned}$$

where $B_{ab}(\Delta) \equiv B^{c}{}_{acb}(\Delta)$ are the components of the symmetric Ricci tensor and $B(\Delta) \equiv B^{a}{}_{a}(\Delta)$ is the Ricci scalar. Note that $K_{abc} \equiv g_{ad} \, K^{d}_{bc}$ and $K^{acb} \equiv g^{cd} \, g^{be} \, K^{a}_{de}$.

Now since

$$\Gamma^b_{ba} = \Delta^b_{ba} = \Delta^b_{ab} = \partial_a \left(\ln \sqrt{\det(g)}\right)$$
$$\Gamma^b_{ab} = \partial_a \left(\ln \sqrt{\det(g)}\right) + 2\,\Gamma^b_{[ab]}$$

we see that for a continuous metric determinant, the so-called homothetic curvature vanishes:

$$H_{ab} \equiv R^c{}_{cab} = \partial_a \Gamma^c_{cb} - \partial_b \Gamma^c_{ca} = 0$$

Introducing the traceless Weyl tensor C, we have the following decomposition theorem:

$$R^d{}_{abc} = C^d{}_{abc} + \frac{1}{n-2}\left(\delta^d_b\ R_{ac} + g_{ac}\ R^d_{\ b} - \delta^d_c\ R_{ab} - g_{ab}\ R^d_{\ c}\right)$$
$$+ \frac{1}{(n-1)(n-2)}\left(\delta^d_c\ g_{ab} - \delta^d_b\ g_{ac}\right) R$$

which is valid for $n > 2$. For $n = 2$, we have

$$R^d{}_{abc} = K_G\left(\delta^d_b\ g_{ac} - \delta^d_c\ g_{ab}\right)$$

where

$$K_G = \frac{1}{2} R$$

is the Gaussian curvature of the surface. Note that (in this case) the Weyl tensor vanishes.

A $n-$dimensional manifold (for which $n > 1$) with constant sectional curvature R and vanishing torsion is called an Einstein space. It is described by

$$R^d{}_{abc} = \frac{1}{n(n-1)}\left(\delta^d_b\ g_{ac} - \delta^d_c\ g_{ab}\right) R$$
$$R_{ab} = \frac{1}{n}\, g_{ab}\ R$$

In the above, we note especially that

$$R^d{}_{abc} = B^d{}_{abc}(\Delta)$$
$$R_{ab} = B_{ab}(\Delta)$$
$$R = B(\Delta)$$

Furthermore, after some elaborate (if not tedious) algebra, we obtain, in general, the following *generalized* Bianchi identities:

$$R^{a}{}_{bcd} + R^{a}{}_{cdb} + R^{a}{}_{dbc} = -2\left(\partial_d \Gamma^{a}_{[bc]} + \partial_b \Gamma^{a}_{[cd]} + \partial_c \Gamma^{a}_{[db]} + \Gamma^{a}_{eb}\,\Gamma^{e}_{[cd]} + \Gamma^{a}_{ec}\,\Gamma^{e}_{[db]} + \Gamma^{a}_{ed}\,\Gamma^{e}_{[bc]}\right)$$
$$\nabla_e R^{a}{}_{bcd} + \nabla_c R^{a}{}_{bde} + \nabla_d R^{a}{}_{bec} = 2\left(\Gamma^{f}_{[cd]}\, R^{a}{}_{bfe} + \Gamma^{f}_{[de]}\, R^{a}{}_{bfc} + \Gamma^{f}_{[ec]}\, R^{a}{}_{bfd}\right)$$
$$\nabla_a\left(R^{ab} - \frac{1}{2}\, g^{ab}\, R\right) = 2\, g^{ab}\, \Gamma^{c}_{[da]}\, R^{d}_{c} + \Gamma^{b}_{[cd]}\, R^{cda}{}_{b}$$

for any metric-compatible manifold endowed with both curvature and torsion.

In the last of the above set of equations, we have introduced the generalized Einstein tensor, i.e.,

$$G_{ab} \equiv R_{ab} - \frac{1}{2}\, g_{ab}\, R$$

In particular, we also have the following specialized identities, i.e., the *regular* Bianchi identities:

$$B^{a}{}_{bcd} + B^{a}{}_{cdb} + B^{a}{}_{dbc} = 0$$
$$\hat{\nabla}_e B^{a}{}_{bcd} + \hat{\nabla}_c B^{a}{}_{bde} + \hat{\nabla}_d B^{a}{}_{bec} = 0$$
$$\hat{\nabla}_a\left(B^{ab} - \frac{1}{2}\, g^{ab}\, B\right) = 0$$

In general, these hold in the case of a symmetric, metric-compatible connection.

4. The structural equations

The results of the preceding section can be expressed in the language of exterior calculus in a somewhat more compact form.

In general, we can construct arbitrary $p-$forms $\omega^{ab\ldots e}_{cd\ldots f}$ through arbitrary $(p-1)$ forms $\alpha^{ab\ldots e}_{cd\ldots f}$, i.e.,

$$\omega^{ab\ldots e}_{cd\ldots f} = d\alpha^{ab\ldots e}_{cd\ldots f} = \frac{\partial\,\alpha^{ab\ldots e}_{cd\ldots f}}{\partial x^h} \wedge dx^h$$

The covariant exterior derivative is then given by

$$D\omega^{ab\ldots e}_{cd\ldots f} = \nabla_h \, \omega^{ab\ldots e}_{cd\ldots f} \wedge dx^h$$

i.e.,

$$\begin{aligned} D\omega^{ab\ldots e}_{cd\ldots f} = d\omega^{ab\ldots e}_{cd\ldots f} + (-1)^p \Big(& \omega^{hb\ldots e}_{cd\ldots f} \wedge \Gamma^a_h + \omega^{ah\ldots e}_{cd\ldots f} \wedge \Gamma^b_h + \ldots + \omega^{ab\ldots h}_{cd\ldots f} \wedge \Gamma^e_h \\ & - \omega^{ab\ldots e}_{hd\ldots f} \wedge \Gamma^h_c - \omega^{ab\ldots e}_{ch\ldots f} \wedge \Gamma^h_d - \ldots - \omega^{ab\ldots e}_{cd\ldots h} \wedge \Gamma^h_f \Big) \end{aligned}$$

where we have defined the connection one-forms by

$$\Gamma^a_b \equiv \Gamma^a_{bc} \, \theta^c$$

with respect to the coframe basis θ^a.

Now we write the torsion two-forms τ^a as

$$\tau^a = D\theta^a = d\theta^a + \Gamma^a_b \wedge \theta^b$$

This gives the first structural equation. With respect to another local coordinate system (with coordinates $\bar{x}^\alpha$) in C^∞ spanned by the basis $\varepsilon^\alpha = e^\alpha_a \, \theta^a$, we see that

$$\tau^a = -\, e^a_\alpha \, \overline{\Gamma}^\alpha_{[\beta\lambda]} \, \varepsilon^\beta \wedge \varepsilon^\lambda$$

We shall again proceed to define the curvature tensor. For a triad of arbitrary vectors u, v, w, we may define the following relations with respect to the frame basis ω_a:

$$\begin{aligned} \nabla_u \nabla_v \, w &\equiv u^c \, \nabla_c \left(v^b \, \nabla_b \, w^a \right) \omega_a \\ \nabla_{[u,v]} \, w &\equiv \nabla_b \, w^a \left(u^c \, \nabla_c \, v^b - v^c \, \nabla_c \, u^b \right) \end{aligned}$$

where ∇_u and ∇_v denote covariant differentiation in the direction of u and of v, respectively.

Then we have

$$\left(\nabla_u \nabla_v - \nabla_v \nabla_u \right) w = {}^*R^a{}_{bcd} \; w^b \; u^c \; v^d \; \omega_a$$

Note that

$$\begin{aligned} {}^*R^a{}_{bcd} &= \partial_c \, \Gamma^a_{bd} - \partial_d \, \Gamma^a_{bc} + \Gamma^e_{bd} \, \Gamma^a_{ec} - \Gamma^e_{bc} \, \Gamma^a_{ed} + 2 \, \Gamma^e_{[cd]} \, \Gamma^a_{be} \\ &= R^a{}_{bcd} + 2 \, \Gamma^e_{[cd]} \, \Gamma^a_{be} \end{aligned}$$

are the components of the extended curvature tensor ${}^*R$.

Define the curvature two-forms by

$$ {}^*R^a_{\ b} \equiv \frac{1}{2}\, {}^*R^a_{\ bcd}\ \theta^c \wedge \theta^d $$

The second structural equation is then

$$ {}^*R^a_{\ b} = d\Gamma^a_b + \Gamma^a_c \wedge \Gamma^c_b $$

The third structural equation is given by

$$ d^2\Gamma^a_b = d\,{}^*R^a_b - {}^*R^a_{\ c} \wedge \Gamma^c_b + \Gamma^a_c \wedge {}^*R^c_{\ b} = D\,{}^*R^a_{\ b} $$

which is equivalent to the generalized Bianchi identities given in the preceding section.

In fact the second and third structural equations above can be directly verified using the properties of exterior differentiation given in Section 2.

Now, as we have seen, the covariant exterior derivative of arbitrary one-forms ϕ^a is given by $D\phi^a = d\phi^a + \Gamma^a_b \wedge \phi^b$. Then

$$ \begin{aligned} DD\phi^a &= d\left(D\phi^a\right) + \Gamma^a_b \wedge D\phi^b \\ &= d\left(d\phi^a + \Gamma^a_b \wedge \phi^b\right) + \Gamma^a_c \wedge \left(d\phi^c + \Gamma^c_d \wedge \phi^d\right) \\ &= d\Gamma^a_b \wedge \phi^b - \Gamma^a_b \wedge \Gamma^b_c \wedge \phi^c \\ &= \left(d\Gamma^a_b + \Gamma^a_c \wedge \Gamma^c_b\right) \wedge \phi^b \end{aligned} $$

where we have used the fact that the $D\phi^a$ are two-forms. Therefore, from the second structural equation, we have

$$ DD\phi^a = {}^*R^a_{\ b} \wedge \phi^b $$

Finally, taking $\phi^a = \theta^a$, we give the fourth structural equation as

$$ DD\theta^a = D\tau^a = {}^*R^a_{\ b} \wedge \theta^b $$

or,

$$ d\tau^a = {}^*R^a_{\ b} \wedge \theta^b - \Gamma^a_b \wedge \tau^b $$

Remarkably, this is equivalent to the first generalized Bianchi identity given in the preceding section.

The elegant results in this section are especially due to E. Cartan. Thanks to his intuitive genius!

5. The geometry of distant parallelism

Let us now consider a special situation in which our $n-$dimensional manifold C^∞ is embedded *isometrically* in a flat $n-$dimensional (pseudo-)Euclidean space E^n (with coordinates $v^{\overline{m}}$) spanned by the constant basis $e_{\overline{m}}$ whose dual is denoted by $s^{\overline{n}}$. This embedding allows us to globally cover the manifold C^∞ in the sense that its geometric structure can be parametrized by the Euclidean basis $e_{\overline{m}}$ satisfying

$$\eta_{\overline{m}\overline{n}} = \langle e_{\overline{m}}, e_{\overline{n}} \rangle = diag\left(\pm 1, \pm 1, ..., \pm 1\right)$$

It is important to note that this situation is different from the one presented in Section 1, in which case we may refer the structural equations of C^∞ to the locally flat tangent space $T_x(M)$. The results of the latter situation (i.e., the *localized* structural equations) should not always be regarded as globally valid since the tangent space $T_x(M)$, though ubiquitous in the sense that it can be defined everywhere (at any point) in C^∞, cannot cover the whole structure of the curved manifold C^∞ without changing orientation from point to point.

One can construct geometries with special connections that will give rise to what we call *geometries with parallelism.* Among others, the geometry of *distant parallelism* is a famous case. Indeed, A. Einstein adopted this geometry in one of his attempts to geometrize physics, and especially to unify gravity and electromagnetism. In its application to physical situations, the resulting field equations of a unified field theory based on distant parallelism, for instance, are quite remarkable in that the so-called energy-momentum tensor appears to be geometrized via the torsion tensor. We will therefore dedicate this section to a brief presentation of the geometry of distant parallelism in the language of Riemann-Cartan geometry.

In this geometry, it is possible to orient vectors such that their directions remain invariant after being displaced from a point to some distant point in the manifold. This situation is made possible by the vanishing of the curvature tensor, which is given by the integrability condition

$$R^d{}_{abc} = e^d_{\overline{m}} \left(\partial_b \partial_c - \partial_c \partial_b\right) e^{\overline{m}}_a = 0$$

where the connection is now given by

$$\Gamma^{c}_{ab} = e^{c}_{\overline{m}}\ \partial_b\, e^{\overline{m}}_{a}$$

where $e^{\overline{m}}_{a} = \partial_a\, \xi^{\overline{m}}$ and $e^{a}_{\overline{m}} = \partial_{\overline{m}}\, x^{a}$.

However, while the curvature tensor vanishes, one still has the torsion tensor given by

$$\Gamma^{a}_{[bc]} = \frac{1}{2}\, e^{a}_{\overline{m}}\, \left(\partial_c\, e^{\overline{m}}_{b} - \partial_b\, e^{\overline{m}}_{c}\right)$$

with the $e^{\overline{m}}_{a}$ acting as the components of a spin "potential". Thus the torsion can now be considered as the primary geometric object in the manifold C^{∞}_{p} endowed with distant parallelism.

Also, in general, the Riemann-Christoffel curvature tensor is non-vanishing as

$$B^{d}{}_{abc} = \hat{\nabla}_c\, K^{d}_{ab} - \hat{\nabla}_b\, K^{d}_{ac} + K^{e}_{ab}\, K^{d}_{ec} - K^{e}_{ac}\, K^{d}_{eb}$$

Let us now consider some facts. Taking the covariant derivative of the tetrad $e^{\overline{m}}_{a}$ with respect to the Christoffel symbols alone, we have

$$\hat{\nabla}_b\, e^{\overline{m}}_{a} = \partial_b\, e^{\overline{m}}_{a} - e^{\overline{m}}_{d}\ \Delta^{d}_{ab} = e^{\overline{m}}_{c}\ K^{c}_{ab}$$

i.e.,

$$K^{c}_{ab} = e^{c}_{\overline{m}}\ \hat{\nabla}_b\, e^{\overline{m}}_{a} = -\, e^{\overline{m}}_{a}\ \hat{\nabla}_b\, e^{c}_{\overline{m}}$$

In the above sense, the components of the contorsion tensor give the so-called *Ricci rotation coefficients*. Then from

$$\hat{\nabla}_c\, \hat{\nabla}_b\, e^{\overline{m}}_{a} = e^{\overline{m}}_{d}\, \left(\hat{\nabla}_c\, K^{d}_{ab} + K^{e}_{ab}\ K^{d}_{ec}\right)$$

it is elementary to show that

$$\left(\hat{\nabla}_c\, \hat{\nabla}_b - \hat{\nabla}_b\, \hat{\nabla}_c\right) e^{\overline{m}}_{a} = e^{\overline{m}}_{d}\ B^{d}{}_{abc}$$

Likewise, we have

$$\tilde{\nabla}_b\, e^{\overline{m}}_{a} = \partial_b\, e^{\overline{m}}_{a} - e^{\overline{m}}_{d}\ K^{d}_{ab} = e^{\overline{m}}_{c}\ \Delta^{c}_{ab}$$
$$\Delta^{c}_{ab} = e^{c}_{\overline{m}}\ \tilde{\nabla}_b\, e^{\overline{m}}_{a} = -\, e^{\overline{m}}_{a}\ \tilde{\nabla}_b\, e^{c}_{\overline{m}}$$

where now $\widetilde{\nabla}$ denotes covariant differentiation with respect to the Ricci rotation coefficients alone. Then from

$$\widetilde{\nabla}_c \widetilde{\nabla}_b e_a^{\overline{m}} = e_d^{\overline{m}} \left(\widetilde{\nabla}_c \Delta_{ab}^d + \Delta_{ab}^e \Delta_{ec}^d \right)$$

we get

$$\left(\widetilde{\nabla}_c \widetilde{\nabla}_b - \widetilde{\nabla}_b \widetilde{\nabla}_c \right) e_a^{\overline{m}} = - e_d^{\overline{m}} \left(B^d{}_{abc} - 2 \Delta_{ae}^d \Gamma_{[bc]}^e - \Delta_{ab}^e K_{ec}^d + \Delta_{ac}^e K_{eb}^d - K_{ab}^e \Delta_{ec}^d + K_{ac}^e \Delta_{eb}^d \right)$$

In this situation, one sees, with respect to the coframe basis $\theta^a = e_{\overline{m}}^a s^{\overline{m}}$, that

$$d\theta^a = - \Gamma_b^a \wedge \theta^b \equiv T^a$$

i.e.,

$$T^a = \Gamma_{[bc]}^a \theta^b \wedge \theta^c$$

Thus the torsion two-forms of this geometry are now given by T^a (instead of τ^a of the preceding section). We then realize that

$$D\theta^a = 0$$

Next, we see that

$$\begin{aligned} d^2\theta^a &= dT^a = - d\Gamma_b^a \wedge \theta^b + \Gamma_b^a \wedge d\theta^b \\ &= - \left(d\Gamma_b^a + \Gamma_c^a \wedge \Gamma_b^c \right) \wedge \theta^b \\ &= - {}^*R_b^a \wedge \theta^b \end{aligned}$$

But, as always, $d^2\theta^a = 0$, and therefore we have

$${}^*R_b^a \wedge \theta^b = 0$$

Note that in this case, ${}^*R_b^a \neq 0$ as

$${}^*R^a{}_{bcd} = 2 \Gamma_{[cd]}^e \Gamma_{be}^a$$

will not vanish in general. We therefore see immediately that

$${}^*R^a{}_{bcd} + {}^*R^a{}_{cdb} + {}^*R^a{}_{dbc} = 0$$

giving the integrability condition

$$\Gamma^e_{[cd]}\, \Gamma^a_{be} + \Gamma^e_{[db]}\, \Gamma^a_{ce} + \Gamma^e_{[bc]}\, \Gamma^a_{de} = 0$$

Meanwhile, the condition

$$dT^a = 0$$

gives the integrability condition

$$\partial_d\, \Gamma^a_{[bc]} + \partial_b\, \Gamma^a_{[cd]} + \partial_c\, \Gamma^a_{[db]} = 0$$

Contracting, we find

$$\partial_c\, \Gamma^c_{[ab]} = 0$$

It is a curious fact that the last two relations somehow remind us of the algebraic structure of the components of the electromagnetic field tensor in physics.

Finally, from the contraction of the components $B^d{}_{abc}$ of the Riemann-Christoffel curvature tensor (the Ricci tensor), one defines the regular Einstein tensor by

$$\hat{G}_{ab} \equiv B_{ab} - \frac{1}{2}\, g_{ab}\, B \equiv k\, E_{ab}$$

where k is a physical coupling constant and E_{ab} are the components of the so-called energy-momentum tensor. We therefore see that

$$\begin{aligned} E_{ab} = \frac{1}{k} & \left(\hat{\nabla}_b\, K^c_{ac} - \hat{\nabla}_c\, K^c_{ab} + K^d_{ac}\, K^c_{db} - K^d_{ab}\, K^c_{dc}\right) \\ & - \frac{1}{2k}\, g_{ab} \left(2\, g^{cd}\, \hat{\nabla}_c\, \Gamma^e_{[de]} + g^{ce}\, \Gamma^d_{[cd]}\, \Gamma^f_{[ef]} + K_{cde}\, K^{ced} - 2\, g^{cd}\, \Gamma^f_{[ed]}\, \Gamma^e_{[cf]}\right) \end{aligned}$$

In addition, the following two conditions are satisfied:

$$E_{[ab]} = 0$$
$$\hat{\nabla}_a\, E^{ab} = 0$$

We have now seen that, in this approach, the energy-momentum tensor (matter field) is fully geometrized. This way, gravity arises from torsional (spin) interaction (possibly on the microscopic scales) and is therefore an *emergent* phenomenon rather than a fundamental one. This seems rather speculative. However, it has profound consequences.

6. Spin frames

A spin frame is described by the anti-symmetric tensor product

$$\Omega^{ik} = \frac{1}{2}\left(\theta^i \otimes \theta^k - \theta^k \otimes \theta \right) = \theta^i \wedge \theta^k \equiv \frac{1}{2}\left[\theta^i , \theta^k\right]$$

In general, then, for arbitrary vector field fields A and B, we can form the commutator

$$\left[A, B\right] = A \otimes B - B \otimes A$$

Introducing another vector field C, we have the so-called Jacobi identity

$$\left[A, \left[B, C\right]\right] + \left[B, \left[C, A\right]\right] + \left[C, \left[A, B\right]\right] = 0$$

With respect to the local coordinate basis elements $E_i = \partial_i$ of the tangent space $T_x(M)$, we see that, astonishingly enough, the anti-symmetric product $\left[A, B\right]$ is what defines the Lie (exterior) derivative of B with respect to A:

$$L_A B \equiv \left[A, B\right] = \left(A^i\ \partial_i B^k - B^i\ \partial_i A^k\right) \frac{\partial}{\partial X^k}$$

(Note that $L_A A = \left[A, A\right] = 0$.) The terms in the round brackets are just the components of our Lie derivative which can be used to define a diffeomorphism invariant (i.e., by taking $A^i = \xi^i$ where ξ represents the displacement field in a neighborhood of coordinate points).

Furthermore, for a vector field U and a tensor field T, both arbitrary, we have (in component notation) the following:

$$\begin{aligned} L_U T^{ij\ldots s}_{kl\ldots r} &= \partial_m T^{ij\ldots s}_{kl\ldots r}\ U^m + T^{ij\ldots s}_{ml\ldots r}\ \partial_k U^m + T^{ij\ldots s}_{km\ldots r}\ \partial_l U^m + \ldots + T^{ij\ldots s}_{kl\ldots m}\ \partial_r U^m \\ &- T^{mj\ldots s}_{kl\ldots r}\ \partial_m U^i - T^{im\ldots s}_{kl\ldots r}\ \partial_m U^j - \ldots - T^{ij\ldots m}_{kl\ldots r}\ \partial_m U^s \end{aligned}$$

It is not immediately apparent whether these transform as components of a tensor field or not. However, with the help of the torsion tensor and the relation

$$\partial_k U^i = \nabla_k U^i - \Gamma^i_{mk}\ U^m = \nabla_k U^i - \left(\Gamma^i_{km} - 2\ \Gamma^i_{[km]}\right) U^m$$

we can write

$$\begin{aligned} L_U\, T^{ij\ldots s}_{kl\ldots r} &= \nabla_m\, T^{ij\ldots s}_{kl\ldots r}\; U^m + T^{ij\ldots s}_{ml\ldots r}\; \nabla_k\, U^m + T^{ij\ldots s}_{km\ldots r}\; \nabla_l\, U^m + \ldots + T^{ij\ldots s}_{kl\ldots m}\; \nabla_r\, U^m \\ &\quad - T^{mj\ldots s}_{kl\ldots r}\; \nabla_m\, U^i - T^{im\ldots s}_{kl\ldots r}\; \nabla_m\, U^j - \ldots - T^{ij\ldots m}_{kl\ldots r}\; \nabla_m\, U^s \\ &\quad + 2\, \Gamma^i_{[mp]}\; T^{mj\ldots s}_{kl\ldots r}\; U^p + 2\, \Gamma^j_{[mp]}\; T^{im\ldots s}_{kl\ldots r}\; U^p + \ldots + 2\, \Gamma^s_{[mp]}\; T^{ij\ldots m}_{kl\ldots r}\; U^p \\ &\quad - 2\, \Gamma^m_{[kp]}\; T^{ij\ldots s}_{ml\ldots r}\; U^p - 2\, \Gamma^m_{[lp]}\; T^{ij\ldots s}_{km\ldots r}\; U^p - \ldots - 2\, \Gamma^m_{[rp]}\; T^{ij\ldots s}_{kl\ldots m}\; U^p \end{aligned}$$

Hence, noting that the components of the torsion tensor, namely, $\Gamma^i_{[kl]}$, indeed transform as components of a tensor field, it is seen that the $L_U\, T^{ij\ldots s}_{kl\ldots r}$ do transform as components of a tensor field. Apparently, the beautiful property of the Lie derivative (applied to an arbitrary tensor field) is that it is connection-independent even in a curved manifold.

If we now apply the commutator to the frame basis of the base manifold C^∞ itself, we see that (for simplicity, we again refer to the coordinate basis of the tangent space $T_x(M)$)

$$[\omega_a, \omega_b] = \left(\partial_a X^i\; \partial_i \partial_b X^k - \partial_b X^i\; \partial_i \partial_a X^k\right) \frac{\partial}{\partial X^k}$$

Again, writing the tetrads simply as $e^i_a = \partial_a X^i$, $e^a_i = \partial_i x^a$, we have

$$[\omega_a, \omega_b] = \left(\partial_a e^k_b - \partial_b e^k_a\right) \frac{\partial}{\partial X^k}$$

i.e.,

$$[\omega_a, \omega_b] = -\, 2\, \Gamma^c_{[ab]}\; \omega_c$$

Therefore, in the present formalism, the components of the torsion tensor are by themselves proportional to the so-called *structure constants* Ψ^c_{ab} of our rotation group:

$$\Psi^c_{ab} = -\, 2\, \Gamma^c_{[ab]} = -\, e^c_i\, \left(\partial_a e^i_b - \partial_b e^i_a\right)$$

As before, here the tetrad represents a spin potential.

Also note that

$$\Psi^d_{ab}\; \Psi^e_{dc} + \Psi^d_{bc}\; \Psi^e_{da} + \Psi^d_{ca}\; \Psi^e_{db} = 0$$

We therefore observe that, as a consequence of the present formalism of differential geometry, spin fields (*objects of anholonomicity*) in the manifold C^∞ are generated *directly* by the torsion tensor.

7. A semi-symmetric connection based on a semi-simple transitive rotation group

Let us now work in *four* dimensions (since this number of dimensions is most relevant to physics). For a semi-simple transitive rotation group, we can show that

$$[\omega_a, \omega_b] = -\gamma \in_{abcd} \varphi^c \theta^d$$

where $\in_{abcd} = \sqrt{\det(g)}\, \varepsilon_{abcd}$ are the components of the completely anti-symmetric four-dimensional Levi-Civita permutation tensor and φ is a vector field normal to a three-dimensional space (hypersurface) $\sum(t)$ defined as the time section $t = const.$ of C^∞ with local coordinates z^A in C^∞. It satisfies $\varphi_a \varphi^a = \gamma = \pm 1$ and is given by

$$\varphi_a = \frac{1}{6} \gamma \in_{abcd} \in^{ABC} \lambda_A^b \lambda_B^c \lambda_C^d$$

where

$$\lambda_A^a \equiv \partial_A x^a \ , \ \lambda_a^A \equiv \partial_a z^A$$
$$\lambda_A^b \lambda_a^A = \delta_a^b - \gamma \varphi_a \varphi^b$$
$$\lambda_A^a \lambda_a^B = \delta_A^B$$

More specifically,

$$\in_{ABC} \varphi_d = \in_{abcd} \lambda_A^a \lambda_B^b \lambda_C^c$$

from which we find

$$\in_{abcd} = \in_{ABC} \lambda_a^A \lambda_b^B \lambda_c^C \varphi_d + \Lambda_{abcd}$$

where

$$\Lambda_{abcd} = \gamma \left(\in_{ebcd} \varphi_a + \in_{aecd} \varphi_b + \in_{abed} \varphi_c \right) \varphi^e$$

Noting that $\Lambda_{abcd} \varphi^d = 0$, we can define a completely anti-symmetric, three-index, four-dimensional "permutation" tensor by

$$\Phi_{abc} \equiv \in_{abcd} \varphi^d = \gamma \in_{ABC} \lambda_a^A \lambda_b^B \lambda_c^C$$

Obviously, the hypersurface $\sum(t)$ can be thought of as representing the position of a material body at any time t. As such, it acts as a boundary of the so-called world tube of the world lines covering an arbitrary four-dimensional region in C^{∞}.

Meanwhile, in the most general four-dimensional case, the torsion tensor can be decomposed according to

$$\begin{aligned}
&\Gamma^{c}_{[ab]} = \frac{1}{3}\left(\delta^{c}_{b}\ \Gamma^{d}_{[ad]} - \delta^{c}_{a}\ \Gamma^{d}_{[bd]}\right) + \frac{1}{6} \in^{c}{}_{abd} \in^{d}{}_{pqr}\ g^{qs}\ g^{rt}\ \Gamma^{p}_{[st]} + g^{cd}\ Q_{dab} \\
&Q_{abc} + Q_{bca} + Q_{cab} = 0 \\
&Q^{a}{}_{ab} = Q^{a}{}_{ba} = 0
\end{aligned}$$

In our special case, the torsion tensor becomes completely anti-symmetric (in its three indices) as

$$\Gamma^{c}_{[ab]} = -\frac{1}{2}\,\gamma\ g^{ce} \in_{abed}\ \varphi^{d}$$

from which we can write

$$\varphi^{a} = -\frac{1}{3} \in^{abcd}\ \Gamma_{b[cd]}$$

where, as usual, $\Gamma_{b[cd]} = g_{be}\ \Gamma^{e}_{[cd]}$. Therefore, at this point, the full connection is given by (with the Christoffel symbols written explicitly)

$$\Gamma^{c}_{ab} = \frac{1}{2}\, g^{cd}\ \left(\partial_{b}\, g_{da} - \partial_{d}\, g_{ab} + \partial_{a}\, g_{bd}\right) - \frac{1}{2}\,\gamma \in^{c}{}_{abd}\ \varphi^{d}$$

We shall call this special connection "*semi-symmetric*". This gives the following simple conditions:

$$\begin{aligned}
&\Gamma^{c}_{(ab)} = \Delta^{c}_{ab} = \frac{1}{2}\, g^{cd}\ \left(\partial_{b}\, g_{da} - \partial_{d}\, g_{ab} + \partial_{a}\, g_{bd}\right) \\
&K^{c}_{ab} = \Gamma^{c}_{[ab]} \\
&\Gamma^{b}_{[ab]} = 0 \\
&\Gamma^{b}_{ab} = \Gamma^{b}_{ba} = \partial_{a}\left(\ln\sqrt{\det(g)}\right)
\end{aligned}$$

Furthermore, we can extract a projective metric tensor ϖ from the torsion (via the structure constants) as follows:

$$\varpi_{ab} = g_{ab} - \gamma \ \varphi_a \ \varphi_b = 2 \ \Gamma^c_{[ad]} \ \Gamma^d_{[cb]}$$

In three dimensions, the above relation gives the so-called Cartan metric.

Finally, we are especially interested in how the torsion tensor affects a coordinate frame spanned by the elements of the basis one-form ω_a and its dual θ^b in a geometry endowed with distant parallelism. Taking the *four-dimensional curl* of the coframe basis θ^b, we see that

$$\begin{aligned}\left[\nabla, \theta^a\right] &= 2 \ d\theta^a = 2 \ T^a \\ &= -\gamma \in^{\bar{m}\bar{n}\bar{p}\bar{q}} \left(\partial_{\bar{m}} \ e^a_{\bar{n}}\right) \varphi_{\bar{p}} \ e_{\bar{q}}\end{aligned}$$

where $\nabla = \theta^b \ \nabla_b = s^{\bar{m}} \ \partial_{\bar{m}}$ and $\in^{abcd} = \dfrac{1}{\sqrt{\det(g)}} \ \varepsilon^{abcd}$. From the metricity condition of the tetrad (with respect to the basis of E^n), namely, $\nabla_b \ e^{\bar{m}}_a = 0$, we have

$$\begin{aligned}\partial_b \ e^{\bar{m}}_a &= \Gamma^c_{ab} \ e^{\bar{m}}_c \\ \partial^{\bar{n}} \ e^{\bar{m}}_a &= \eta^{\bar{n}\bar{p}} \ e^b_{\bar{p}} \ \partial_b \ e^{\bar{m}}_a = e^{\bar{m}}_c \ \Gamma^c_{ab} \ e^{\bar{n}b}\end{aligned}$$

It is also worthwhile to note that from an equivalent metricity condition, namely, $\nabla_a \ e^b_{\bar{m}} = 0$, one finds

$$\partial_{\bar{n}} \ e^a_{\bar{m}} = -\Gamma^a_{bc} \ e^b_{\bar{m}} \ e^c_{\bar{n}}$$

Thus we find

$$\left[\nabla, \theta^a\right] = -\gamma \in^{bcde} \Gamma^a_{[bc]} \ \varphi_d \ \omega_e$$

In other words,

$$T^a = d\theta^a = -\frac{1}{2} \gamma \in^{bcde} \Gamma^a_{[bc]} \ \varphi_d \ \omega_e$$

For the frame basis, we have

$$\left[\nabla, \omega_a\right] = -\gamma \in^{bcde} \Gamma_{a[bc]} \ \varphi_d \ \omega_e$$

At this point it becomes clear that the presence of torsion in C^∞ rotates the frame and coframe bases themselves. The basics presented here constitute the reality of the so-called *spinning frames*.

Suggested general references

In writing this article, I have had to rely on my own intuition (mental construct), understanding, and memory alone without any particular attachment to just one or two of the well-known works. As for references or further reading, there's a number of widely appreciated books and articles on differential geometry. I myself have come across some of them. Their in-depth presentation of the subject provides excellent educational material. Also, I feel that it is important to get a sense of history and so reading the writings of the founding fathers of (modern) differential geometry (especially Cartan's writings) is essential. One may compare the present work to the following general references (most of them, especially the later ones, are more advanced):

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