The easiest way to approach this analysis is always to calculate the acceleration firstly in plane polar coordinates and to realize that one term of the resultant expression is the acceleration in the Cartesian system. For an observer on earth orbiting the sun, the relevant expression is that in the Cartesian frame, because the latter is also fixed on the earth and does not move with respect to the observer. In other words the observer is in his own frame of reference. For an observer on the sun the relevant expression is that in the plane polar system of coordinates, because the earth rotates with respect to the observer fixed on the sun.

The observer on the earth experiences the centrifugal acceleration:

$$-\underline{\omega} \times (\underline{\omega} \times \underline{r}) = \omega^2 r \, \underline{e}_r \qquad -(165)$$

directed outwards from the earth. This is the origin of the everyday centrifugal force. The observer on the sun experiences the centripetal acceleration:

$$\underline{\omega} \times (\underline{\omega} \times \underline{r}) = -\omega^2 r \, \underline{e}_r \qquad -(166)$$

directed towards the sun and towards the observer. The entire analysis rests on the spin connection and on the fact that in the plane polar system the frame itself is rotating and thus generates the spin connection by definition.

8.4 DESCRIPTION OF ORBITS WITH THE MINKOWSKI FORCE EQUATION.

In UFT 238 on www.aias.us an entirely new approach to orbital theory was taken using the Minkowski force equation. This is a course that relativity theory could have taken, but cosmology followed the use of Einstein's flawed geometry, a subject that became known as general relativity. The Minkowski force equation is the Newton force equation with proper time τ replacing time t. This equation was inferred by Minkowski shortly after Einstein's introduction of the idea of relativistic momentum. A completely general kinematic theory of orbits can be developed in this way. It reduces to the Newtonian theory but never to the Einsteinian theory. Newtonian dynamics does not give any of the forces that are generated as discussed in Section 8.3 using plane polar coordinates and a system of rotating coordinates. It turns out that the space part of the Minkowski four force produces new and unexpected orbital properties that can be tested experimentally.

The relativistic force law and relativistic orbits of the Minkowski equation can be derived by considering the relativistic velocity in plane polar coordinates:

$$\underline{v} = \frac{d\underline{r}}{d\tau} = \gamma \frac{d\underline{r}}{dt} \quad -(167)$$

where τ is the proper time and γ the Lorentz factor:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(168)$$

$$-(169)$$

The relativistic acceleration is:

$$\underline{a} = \frac{d}{d\tau}\left(\frac{d\underline{r}}{d\tau}\right) = \frac{d}{d\tau}\left(\gamma \frac{d\underline{r}}{dt}\right) = \gamma \frac{d}{dt}\left(\gamma \frac{d\underline{r}}{dt}\right).$$

Using the Leibnitz Thoerem:

$$\underline{a} = \gamma \left(\frac{d\gamma}{dt} \frac{d\underline{r}}{dt} + \gamma \frac{d}{dt}\left(\frac{d\underline{r}}{dt}\right)\right). \quad -(170)$$

The velocity v appearing in the Lorentz factor is defined by the infinitesimal line element:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - d\underline{r} \cdot d\underline{r} \quad -(171)$$

where:

$$d\underline{r} \cdot d\underline{r} = v^2 dt^2 \quad -(172)$$

Therefore

$$c^2 d\tau^2 = (c^2 - v^2)\, dt^2 \quad -(173)$$

and the Lorentz factor is:

$$\gamma = \frac{dt}{d\tau} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \quad -(174)$$

In plane polar coordinates:

$$d\underline{r} \cdot d\underline{r} = dr^2 + r^2 d\theta^2. \quad -(175)$$

Therefore:

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2. \quad -(176)$$

The radial vector in plane polar coordinates is:

$$\underset{\sim}{r} = r\,\underline{e}_r \quad -(177)$$

therefore the non relativistic velocity is:

$$\underline{v} = \frac{d}{dt}(r\,\underline{e}_r) = \frac{dr}{dt}\underline{e}_r + r\frac{d\underline{e}_r}{dt} = \frac{dr}{dt}\underline{e}_r + \omega r\,\underline{e}_\theta$$

$$= \frac{dr}{dt}\underline{e}_r + \underline{\omega} \times \underline{r} = \left(\frac{L_0}{m}\right)\left(\frac{1}{r}\underline{e}_\theta - \frac{d}{d\theta}\left(\frac{1}{r}\right)\underline{e}_r\right). \quad -(178)$$

For a particle of mass m in an orbit, its relativistic momentum is:

$$\underline{p} = \gamma m \frac{d\underline{r}}{dt} = m\frac{d\underline{r}}{d\tau} \quad -(179)$$

an equation which can be rearranged as follows:

$$p^2c^2 = \gamma^2 m^2 c^4 \left(\frac{v}{c}\right)^2 = \gamma^2 m^2 c^4 \left(1 - \frac{1}{\gamma^2}\right) = \gamma^2 m^2 c^4 - m^2 c^4 \quad -(180)$$

giving the Einstein energy equation:

$$E^2 = c^2 p^2 + m^2 c^4 \quad -(181)$$

in which

$$E = \gamma m c^2 \quad -(182)$$

is the total energy and

$$E_0 = m c^2 \quad -(183)$$

is the rest energy. The relativistic total angular momentum is:

$$L = m r^2 \frac{d\theta}{d\tau} = \gamma L_0 . \quad -(184)$$

The concept of Minkowski force equation uses acceleration, so this is a plausible new approach to all orbits. The Einstein energy equation can be derived from the infinitesimal line element (171) and developed as:

$$mc^2 = mc^2 \left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dr}{d\tau}\right)^2 - r^2 \left(\frac{d\theta}{d\tau}\right)^2$$
$$= \gamma^2 m c^2 - \left(\left(\frac{dr}{d\tau}\right)^2 + r^2 \left(\frac{d\theta}{d\tau}\right)^2\right) = \frac{E^2}{mc^2} - \frac{p^2}{c^2} \quad -(185)$$

So

$$E^2 = c^2 p^2 + m^2 c^4 \quad -(186)$$

Q. E. D. The relativistic linear momentum in Eq. (185) is:

$$p^2 = m^2 \left(\left(\frac{dr}{d\tau}\right)^2 + r^2 \left(\frac{d\theta}{d\tau}\right)^2\right) \quad -(187)$$

which is Eq. (179), Q. E. D. The definition of relativistic acceleration is

$$\underline{a} = \frac{d}{d\tau}\left(\frac{d\underline{r}}{d\tau}\right) = \gamma\left(\frac{d\gamma}{dt}\frac{d\underline{r}}{dt} + \gamma\frac{d}{dt}\left(\frac{d\underline{r}}{dt}\right)\right) \quad -(188)$$

in which:

$$\frac{d\underline{r}}{dt} = \frac{dr}{dt}\underline{e}_r + \underline{\omega} \times \underline{r} \quad -(189)$$

and

$$\frac{d}{dt}\left(\frac{d\underline{r}}{dt}\right) = \frac{d^2 r}{dt^2}\underline{e}_r + \frac{d\underline{\omega}}{dt} \times \underline{r} + 2\underline{\omega} \times \frac{dr}{dt}\underline{e}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad -(190)$$

Using the chain rule:

$$\frac{d\gamma}{dt} = \frac{d\gamma}{dv}\frac{dv}{dt} \quad -(191)$$

where v is the velocity of the Lorentz factor defined in Eq. (174). Therefore:

$$\frac{d\gamma}{dv} = \frac{d}{dv}\left(1 - \frac{v^2}{c^2}\right)^{-1/2} = \gamma^3 \frac{v}{c^2} \quad -(192)$$

and in plane polar coordinates:

$$\underline{a} = \gamma^4 \frac{v}{c^2}\frac{dv}{dt}\frac{d\underline{r}}{dt} + \gamma^2 \frac{d}{dt}\left(\frac{d\underline{r}}{dt}\right) \quad -(193)$$

$$= \left(\frac{d\gamma}{d\tau}\frac{dr}{dt} + \gamma^2\frac{d^2 r}{dt^2}\right)\underline{e}_r + \gamma^2\underline{\omega} \times (\underline{\omega} \times \underline{r}) + \frac{d\gamma}{d\tau}\underline{\omega} \times \underline{r} + \gamma^2\left(\frac{d\underline{\omega}}{dt} \times \underline{r} + 2\underline{\omega} \times \frac{dr}{dt}\underline{e}_r\right)$$

In static Cartesian coordinates on the other hand;

$$\underline{a} = \frac{d}{d\tau}\left(\gamma\frac{d\underline{r}}{dt}\right) = \gamma\frac{d\gamma}{dt}\frac{d\underline{r}}{dt} + \gamma^2\frac{d}{dt}\left(\frac{d\underline{r}}{dt}\right) \quad -(194)$$

so:

$$\underline{a}\left(\text{Cartesian}\right) = \left(\gamma \frac{d\gamma}{dt}\frac{dr}{dt} + \gamma^2 \frac{d^2 r}{dt^2}\right)\underline{e}_r \quad -(195)$$

in which:

$$v = \frac{dr}{dt}, \quad \frac{d^2 r}{dt^2} = \frac{dv}{dt}, \quad \frac{d\gamma}{dv} = \gamma^3 \frac{v}{c^2} \quad -(196)$$

and

$$\frac{d\gamma}{dt} = \frac{d\gamma}{dv}\frac{dv}{dt} = \frac{\gamma^3 v}{c^2}\frac{dv}{dt}. \quad -(197)$$

Therefore:

$$\underline{a}\left(\text{Cartesian}\right) = \left(\gamma^4 \frac{v^2}{c^2} + \gamma^2\right)\frac{dv}{dt}\,\underline{e}_r \quad -(198)$$

in which:

$$\frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2}. \quad -(199)$$

Therefore the Cartesian acceleration is:

$$\underline{a}\left(\text{Cartesian}\right) = \gamma^4 \frac{d^2 r}{dt^2}\,\underline{e}_r. \quad -(200)$$

Using Eq. (200) in Eq. (193):

$$\underline{a}\left(\text{plane polar}\right) = \gamma^4 \frac{d^2 r}{dt^2}\underline{e}_r + \gamma^2 \underline{\omega}\times(\underline{\omega}\times\underline{r}) + \frac{d\gamma}{d\tau}\underline{\omega}\times\underline{r} + \gamma^2\left(\frac{d\underline{\omega}}{dt}\times\underline{r} + 2\underline{\omega}\times\frac{dr}{dt}\underline{e}_r\right) \quad -(201)$$

which is the expression for relativistic acceleration in plane polar coordinates.

It can be proven as follows that the relativistic Coriolis acceleration vanishes for all planar orbits. The general expression for relativistic Coriolis acceleration is:

$$\underline{a}(\text{Coriolis}) = \gamma^2\left(r\frac{d}{dt}\frac{d\theta}{dt} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\underline{e}_\theta \quad -(202)$$

in which the total ~~relativistic~~ non relativistic angular momentum is:

$$L_0 = mr^2\frac{d\theta}{dt}. \quad -(203)$$

It follows that:

$$\frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d}{dt}\left(\frac{L_0}{mr^2}\right) = \frac{d}{dr}\left(\frac{L_0}{mr^2}\right)\frac{dr}{dt} = -\frac{2L_0}{mr^3}\frac{dr}{dt} \quad -(204)$$

so:

$$\underline{a}(\text{Coriolis}) = \left(-\frac{2L_0}{mr^2}\frac{dr}{dt} + \frac{2L_0}{mr^2}\frac{dr}{dt}\right)\underline{e}_\theta = \underline{0} \quad -(205)$$

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Therefore the relativistic acceleration for all planar orbits is:

$$\underline{a} = \gamma^4\frac{d^2r}{dt^2}\underline{e}_r + \gamma^2\underline{\omega}\times(\underline{\omega}\times\underline{r}) + \frac{d\gamma}{d\tau}\underline{\omega}\times\underline{r}. \quad -(206)$$

The relativistic centripetal component of this orbit is:

$$\underline{a}(\text{centripetal}) = \gamma^2\underline{\omega}\times(\underline{\omega}\times\underline{r}) = -\frac{L^2}{m^2r^3}\underline{e}_r. \quad -(207)$$

In Eq. (206):

$$\frac{d\gamma}{d\tau} = \gamma\frac{d\gamma}{dv}\frac{dv}{dt} = \frac{\gamma^4}{c^2}v\frac{dv}{dt} = \frac{\gamma^4}{c^2}\frac{dr}{dt}\frac{d^2r}{dt^2} \quad -(208)$$

and therefore the acceleration becomes:

$$\underline{a} = \gamma^4 \frac{d^2 r}{dt^2} \underline{e}_r - \frac{L^2}{m^2 r^3} \underline{e}_r + \frac{\gamma^4}{c^2} \frac{dr}{dt} \frac{d^2 r}{dt^2} \omega r \underline{e}_\theta \quad -(209)$$

in which the relativistic total angular momentum is

$$L = \gamma L_0 = m r^2 \frac{d\theta}{d\tau} = \gamma m r^2 \omega. \quad -(210)$$

The relativistic force law is therefore the mass m multiplied by the relativistic acceleration:

$$\underline{a} = \left(\gamma^4 \frac{d^2 r}{dt^2} - \frac{L^2}{m^2 r^3} \right) \underline{e}_r + \frac{\gamma^4}{c^2} \frac{dr}{dt} \frac{d^2 r}{dt^2} \underline{\omega} \times \underline{r} \quad -(211)$$

in which:

$$\underline{\omega} \times \underline{r} = \omega r \underline{e}_\theta . \quad -(212)$$

This equation can be transformed into a format where the relativistic force can be calculated from the observation of any planar orbit. The result is the relativistic generalization of Eq, (122).

Consider the relativistic acceleration:

$$\underline{a} = \gamma^4 \frac{d^2 r}{dt^2} \underline{e}_r + \gamma^2 \underline{\omega} \times (\underline{\omega} \times \underline{r}) + \frac{d\gamma}{d\tau} \underline{\omega} \times \underline{r} \quad -(213)$$

in which the relativistic momentum is:

$$\underline{p} = m \frac{d\underline{r}}{d\tau} . \quad -(214)$$

It follows that:

$$\frac{d^2 r}{dt^2} = -\left(\frac{L}{\gamma m r} \right)^2 \frac{d^2}{d\theta^2} \left(\frac{1}{r} \right) \quad -(215)$$

and that:

$$\underline{a} = -\left(\left(\frac{\gamma L}{mr}\right)^2 \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{L^2}{m^2 r^3}\right)\underline{e}_r + \frac{\gamma^4}{c^2}\frac{dr}{dt}\frac{d^2 r}{dt^2}\,\omega r\,\underline{e}_\theta \quad -(216)$$

It also follows as in UFT 238 that:

$$\frac{dr}{dt} = -\frac{L}{m\gamma}\frac{d}{d\theta}\left(\frac{1}{r}\right) \quad -(217)$$

$$-(218)$$

so the required relativistic generalization of Eq. (122) is:

$$\underline{a} = -\left(\frac{L}{mr}\right)^2\left(\gamma^2 \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right)\underline{e}_r + \frac{L^4}{m^4 c^2 r^3}\frac{d}{d\theta}\left(\frac{1}{r}\right)\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)\underline{e}_\theta$$

For the purposes of graphics and animation it is convenient to express the Lorentz factor in terms of the angle θ . The result as derived in UFT 238 is:

$$v^2 = \left(\frac{L_0}{md}\right)^2\left(1 + \epsilon^2 + 2\epsilon\cos\theta\right) \quad -(219)$$

$$-(220)$$

In summary, the relativistic force for any planar orbit is defined by:

$$\underline{F} = -\frac{L^2}{mr^2}\left(\gamma^2 \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right)\underline{e}_r + \frac{L^4}{m^4 c^2 r^3}\frac{d}{d\theta}\left(\frac{1}{r}\right)\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)\underline{e}_\theta$$

in which the Lorentz factor is:

$$\gamma = \left(1 - \left(\frac{L_0}{mc}\right)^2\left(\frac{1}{r^2} + \left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2\right)\right)^{-1/2} \quad -(221)$$

and in which the relativistic total angular momentum is:

$$L = \gamma L_0 = \gamma m r^2 \frac{d\theta}{d\tau} = \gamma m r^2 \omega \quad -(222)$$