The AntiSymmetric Metric

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Introduction

It is well known in differential geometry that the tetrad is defined by:

$$\boxed{V^a = q^a_\mu \, V^\mu} \tag{1}$$

Here V^a is a four-vector defined in the Minkowski spacetime of the tangent bundle at point P to the base manifold. The latter is the general 4-D spacetime in which the vector is defined by V^μ.

The metric tensor used by Einstein in his field theory of gravitation (1915) is (Carroll):

$$g^{(S)}_{\mu\nu} = q^a_\mu \, q^b_\nu \, \eta_{ab} \tag{2}$$

In eqn. (2) η_{ab} is the metric of the tangent bundle. Eqn. (2) defines a symmetric metric $g^{(S)}_{\mu\nu}$, through an inner or dot product of two tetrads.

It is seen in eqn. (1) that there is summation over repeated indices. This is the Einstein convention. One index μ is a subscript (covariant) on the right hand side of eqn. (1). Thus, written out in full eqn. (1) is:

$$V^a = q^a_0 \, V^0 + q^a_1 \, V^1 + q^a_2 \, V^2 + q^a_3 \, V^3 \tag{3}$$

Similarly, eqn. (2) is:

$$g^{(S)}_{\mu\nu} = q^0_\mu \, q^0_\nu \, \eta_{00} + \cdots + q^3_\mu \, q^3_\nu \, \eta_{33} \tag{4}$$

In eqn. (4) it is seen that all possible combinations of a, b are summed.

Another example is given by Einstein in his famous book “The Meaning of Relativity" (Princeton, 1921-1954):

$$g_{\mu\nu}^{(S)}\, g^{\mu\nu(S)} = 4 \tag{5}$$

It is seen that the double summation over μ and ν in eqn. (5) produces a scalar (the number 4). In differential geometry a scalar is a zero-form.

It is seen from the basic and well known definition (2) that is possible to define the wedge product of two tetrads:

$$q_{\mu\nu}^{c} = q_{\mu}^{a} \wedge q_{\nu}^{b} \tag{6}$$

The wedge product is a generalization to any dimension of the vector cross product in 3-D. In eqn. (6) $q_{\mu\nu}^{c}$ is a two-form of differential geometry, i.e. a tensor antisymmetric in μ and ν. It is a vector-valued two-form due to the presence of the index c. This is the antisymmetric metric:

$$g_{\mu\nu}^{c(A)} = q_{\mu\nu}^{c} \tag{7}$$

The antisymmetric metric is part of the more general tensor metric formed by the outer product of two tetrads:

$$g_{\mu\nu}^{ab} = q_{\mu}^{a}\, q_{\nu}^{b} \tag{8}$$

It is seen that the indices μ and ν are always the same on both sides, so can be left out for clarity of presentation (see Carroll).

Thus we obtain:

$$q^{ab} = q^{a}\, q^{b} \tag{9}$$

$$g^{c(A)} = q^{a} \wedge q^{b} \tag{10}$$

$$g^{(S)} = q^{a}\, q^{b}\, \eta_{ab} \tag{11}$$

This notation shows clearly that q^{ab} is a tensor; $g^{c(A)}$ is a vector; $g^{(S)}$ is a scalar. It is well known that any tensor is the sum of a symmetric and antisymmetric component:

$$q^{ab} = q^{ab(S)} + q^{ab(A)} \tag{12}$$

Furthermore, $q^{ab(S)}$ is the sum of an off-diagonal symmetric tensor and a diagonal tensor. The sum of the elements of the diagonal tensor is known as the trace.

Thus, $g^{c(A)}$ is the antisymmetric part of q^{ab}:

$$\boxed{g^{c(A)} = \frac{1}{2}\,\epsilon^{abc}\,q^{ab(A)}} \tag{13}$$

In eqn. (11):

$$\eta_{ab} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{14}$$

thus:

$$\begin{aligned} g^{(S)} &= q^0\,q^0\,\eta_{00} + q^1\,q^1\,\eta_{11} + q^2\,q^2\,\eta_{22} + q^3\,q^3\,\eta_{33} \\ &= q^0\,q^0 - q^1\,q^1 - q^2\,q^2 - q^3\,q^3\,\eta_{33} \end{aligned} \tag{15}$$

and so:

$$\boxed{g^{(S)} = \text{Trace}\,q^{ab}} \tag{16}$$

From eqn. (9), (13) and (16) it is seen that the existence of the antisymmetric metric is implied by the existence of the symmetric metric.

Quod erat demostrandum.

In the notation of eqn. (2.33) of Evans, Chapter 2:

$$\omega_2 = -\frac{1}{2}\,q^{\mu\nu(A)}\,d\,u_\mu \wedge d\,u_\nu \tag{17}$$

From the definition of the wedge product, eqn. (6), eqn. (17) is:

$$\omega_2 = -\frac{1}{2}\,q^{\mu\nu(A)}\,q^{(A)}_{\mu\nu} \tag{18}$$

and by comparison with Einstein's eqn. (5), it is seen that ω_2 is a scalar.

Quod erat demostrandum.