OVERALL SUMMARY OF OPTICAL, OR RADIATIVE, AHARONOV BOHM EFFECT

Before Gauge Transformation

$$\underline{B}^{(3)*} = -ig\,\underline{A}^{(1)} \times \underline{A}^{(2)} \quad \text{--- (1)}$$

After Gauge Transformation

$$\underline{B}^{(3)} \longrightarrow \underline{B}^{(3)} \quad \text{--- (2)}$$

$$\underline{A}^{(1)} \times \underline{A}^{(2)} \rightarrow \left(\underline{A}^{(1)} + \underline{a}^{(1)}\right) \times \left(\underline{A}^{(2)} + \underline{a}^{(2)}\right)$$

$$= \underline{A}^{(1)} \times \underline{A}^{(2)} + \underline{A}^{(1)} \times \underline{a}^{(2)} + \underline{a}^{(1)} \times \underline{A}^{(2)} + \underline{a}^{(1)} \times \underline{a}^{(2)} \quad \text{--- (3)}$$

Therefore:

$$\underline{A}^{(1)} \times \underline{a}^{(2)} + \underline{a}^{(1)} \times \underline{A}^{(2)} + \underline{a}^{(1)} \times \underline{a}^{(2)} = \underline{0} \quad \text{--- (4)}$$

The OPTICAL AHARONOV BOHM EFFECT is due to $\underline{a}^{(1)} \times \underline{a}^{(2)}$ in regions where $\underline{A}^{(1)}$ and $\underline{A}^{(2)}$ are not present. For example there is an inverse Faraday effect due to $\underline{a}^{(1)} \times \underline{a}^{(2)}$.

REFERENCES

The Enigmatic Photon, Volume Three and Five.

CHECK ON GAUGE TRANSFORM, $u(1)$

In $u(1)$:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad — (1)$$

$$F_{\mu\nu}' = S F_{\mu\nu} S^{-1} \quad — (2)$$

$$A_\mu \rightarrow A_\mu + \partial_\mu \chi \quad — (3)$$

From (1) and (3):

$$F_{\mu\nu}' = \partial_\mu (A_\nu + \partial_\nu \chi) - \partial_\nu (A_\mu + \partial_\mu \chi)$$
$$= F_{\mu\nu} + (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\chi$$

Therefore: $\boxed{\partial_\mu \partial_\nu \chi = \partial_\nu \partial_\mu \chi}$ — (4)

Therefore the gauge transformed $F_{\mu\nu}'$ is worked out w.r.t. gauge transformed potentials.

This checks that:

$$\underline{A}^{(1)} \times \underline{A}^{(2)} \rightarrow (\underline{A}^{(1)} + \underline{a}^{(1)}) \times (\underline{A}^{(2)} + \underline{a}^{(2)})$$

because:

$$\underline{A}^{(1)} \rightarrow \underline{A}^{(1)} + \underline{a}^{(1)}$$
$$\underline{A}^{(2)} \rightarrow \underline{A}^{(2)} + \underline{a}^{(2)}$$

Quaternions and Spinors

O(3) has the Euler angles α, β, γ. SU(2) is defined by a complex 2-D vector $(\underline{u}, \underline{v})$, and this is rotated by:

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \quad — (1)$$

with determinant $= \pm 1$ and $ad - bc = 1$. In Barrett's notation

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} q_0 + iq_3 & q_1 - iq_2 \\ -q_1 - iq_2 & q_0 - iq_3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \quad — (2)$$

where

$$\left.\begin{aligned} a &:= q_0 + iq_3 = \cos\frac{\beta}{2} \exp\left(\frac{i}{2}(\alpha + \gamma)\right) \\ b &:= q_1 - iq_2 = \sin\frac{\beta}{2} \exp\left(-\frac{i}{2}(\alpha - \gamma)\right) \end{aligned}\right\} \quad — (3)$$

Here

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1 \quad — (4)$$

is the normalization condition on the quaternion components, q_0, q_1, q_2, q_3 is given by:

The O(3) rotation matrix is given by:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} e_{1X} & e_{1Y} & e_{1Z} \\ e_{2X} & e_{2Y} & e_{2Z} \\ e_{3X} & e_{3Y} & e_{3Z} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad — (5)$$

where:

$$e_{1X} = q_0^2 + q_1^2 - q_2^2 - q_3^2 = \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\beta$$

$$e_{1Y} = 2(q_1q_2 + q_0q_3) = \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma$$

$$e_{1Z} = 2(q_1q_3 - q_0q_2) = -\sin\beta\cos\gamma$$

$$e_{2X} = 2(q_1q_2 - q_0q_3) = -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma$$

$$e_{2Y} = q_0^2 - q_1^2 + q_2^2 - q_3^2 = -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma$$

$$e_{2Z} = 2(q_2q_3 + q_0q_1) = \sin\beta\sin\gamma$$

$$e_{3X} = 2(q_1q_3 + q_0q_2) = \cos\alpha\sin\beta$$

$$e_{3Y} = 2(q_2q_3 - q_0q_1) = \sin\alpha\sin\beta$$

$$e_{3Z} = q_0^2 - q_1^2 - q_2^2 + q_3^2 = \cos\beta$$

Rotation About the z Axis

This means $\cos\beta = \cos\gamma = 1$; $\sin\beta = \sin\gamma = 0$, & $\cancel{\alpha \equiv \theta}$

$$l_{1x} = \cos\alpha;\quad l_{1y} = \sin\alpha;\quad l_{1z} = 0$$
$$l_{2x} = -\sin\alpha;\quad l_{2y} = \cos\alpha;\quad l_{2z} = 0$$
$$l_{3x} = 0;\quad l_{3y} = 0;\quad l_{3z} = 1.$$

A possible description of rotation about the z axis is $\beta = 0$, $\gamma = 0$,

i.e. $q_0 = \cos\frac{\alpha}{2}$; $q_3 = \sin\frac{\alpha}{2}$; $q_1 = 0$; $q_2 = 0$.

The rotation matrices are therefore:

$$\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \longleftrightarrow \begin{bmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{bmatrix}$$

$$\begin{bmatrix} q_0^2 - q_3^2 & 2q_0q_3 & 0 \\ -2q_0q_3 & q_0^2 - q_3^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longleftrightarrow \begin{bmatrix} q_0 + iq_3 & 0 \\ 0 & q_0 - iq_3 \end{bmatrix}$$

Therefore we obtain the half angle formulae:

$$\cos\alpha = q_0^2 - q_3^2 = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}$$
$$\sin\alpha = 2q_0q_3 = 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}$$

Notes

1) The o(3) rotation matrix is set up in terms of α and the su(2) in terms of $\alpha/2$.
2) The o(3) rotation matrix is real, the su(2) is complex.
3) The same quaternions appear in o(3) and su(2).

Infinitesimal Rotation Generators in SU(2)

Let:

$$R_\alpha(z) = \begin{bmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{bmatrix}.$$

The infinitesimal rotation generator is:

$$\tau_z := \frac{1}{i} \left.\frac{\partial R_z(\alpha)}{\partial \alpha}\right|_{\alpha=0} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \frac{\sigma_z}{2}.$$

where σ_z is the third Pauli matrix.

We have:

$$e^{i\sigma_z\alpha/2} = 1 + i\sigma_z \frac{\alpha}{2} - \frac{\sigma_z^2}{2!}\frac{\alpha^2}{4} - i\frac{\sigma_z^3}{3!}\frac{\alpha^3}{8} + \cdots\cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\frac{\alpha}{2} - \frac{1}{2!}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\frac{\alpha^2}{4} + \cdots\cdots.$$

$$= \begin{bmatrix} 1 + i\alpha/2 - \alpha^2/4.2! + \cdots & 0 \\ 0 & 1 - i\alpha/2 - \alpha^2/4.2! + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{bmatrix} = \begin{bmatrix} q_0 + iq_3 & 0 \\ 0 & q_0 - iq_3 \end{bmatrix}$$

i.e. $$\boxed{e^{i\sigma_z\alpha/2} = q_0 + iq_3\sigma_z}$$

Small Angle Limit

$$q_0 \to 1\ ;\ q_3 \to \alpha/2$$

$$\boxed{e^{i\sigma_z\alpha/2} \xrightarrow[\alpha\to 0]{} 1 + i\frac{\alpha}{2}\sigma_z}$$

Field Rotation

$$\phi' = e^{i\sigma_Z \alpha/2}\phi$$
$$= (q_0 + i q_3 \sigma_Z)\phi$$

Gauge Transformation

The basis of gauge transformation is that the quaternions become functions of x^μ. This means the introduction of covariant derivatives. Thus:

$$\boxed{\phi'(x^\mu) = (q_0(x^\mu) + i q_3(x^\mu))\phi(x^\mu)}$$

for a field rotation about the Z axis.

Under this transformation, the potential four vector becomes:

$$\boxed{A_\mu' = S A_\mu S^{-1} - \frac{i}{g}\partial_\mu S S^{-1}} \quad \text{—— (1)}$$

where

$$\boxed{\begin{aligned} A_\mu &:= A_\mu^a \sigma^a/2 = A_\mu^Z \sigma^Z/2 \\ S &:= e^{i\sigma_Z \alpha/2} \end{aligned}}$$

Therefore a gauge transformation in SU(2) is the direct consequence of rotating the field ϕ geometrically. Eqn. (1) is the basis of the Aharonov Bohm effect and instanton theory. In SU(2), A_μ is put into spinor form:

$$A_\mu := \begin{bmatrix} A_\mu^Z/2 & 0 \\ 0 & -A_\mu^Z/2 \end{bmatrix} \quad \text{—— (2)}$$

Working out the Gauge Transformed A_μ^Z

We use formula (1) with:

$$S = e^{i\alpha\sigma_z/2} \; ; \; S^{-1} = e^{-i\alpha\sigma_z/2}$$

$$\partial_\mu S = \left(i \frac{\sigma_z}{2} \partial_\mu \alpha \right) S$$

$$A_\mu' = A_\mu - \frac{i^2}{g} \frac{\sigma_z}{2} \partial_\mu \alpha$$

$$\begin{bmatrix} A_\mu^{Z\prime}/2 & 0 \\ 0 & -A_\mu^{Z\prime}/2 \end{bmatrix} = \begin{bmatrix} A_\mu^Z/2 & 0 \\ 0 & -A_\mu^Z/2 \end{bmatrix} + \begin{bmatrix} \partial_\mu\alpha/(2g) & 0 \\ 0 & -\partial_\mu\alpha/(2g) \end{bmatrix}$$

i.e. $$\boxed{A_\mu^{Z\prime} = A_\mu^Z + \frac{1}{g} \partial_\mu \alpha} \longrightarrow (3)$$

where:

$$\alpha = \cos^{-1}(q_0^2 - q_3^2) = \sin^{-1}(2 q_0 q_3)$$

Therefore rotation about Z of the general field ψ implies eqn. (3).

Small Angle Limit

$$\frac{\alpha}{2} \sim \sin\frac{\alpha}{2} = q_3$$

$$A_\mu^{Z\prime} \xrightarrow[\alpha \to 0]{} A_\mu^Z + \frac{1}{g} \frac{\partial q_3}{\partial Z} \quad - (3a)$$

More generally:

$$\boxed{A_\mu^{Z\prime} \xrightarrow[\alpha \to 0]{} A_\mu^Z + \frac{1}{g} \partial_\mu q_3} \quad - (3b)$$

Notes

1) The gauge transform in this view is a geometrical process.
2) If $\partial\alpha / \partial x^\mu = 0$, $A_Z' \to A_Z$. This means that rotation about Z in this case has no effect on A_Z
3) The A_μ is a physical four-potential.

Various Checks of Equation (3)

1) Small Angle rotation, O(3), Ryder p. 119

$$\begin{bmatrix} \phi_1' \\ \phi_2' \\ \phi_3' \end{bmatrix} = \begin{bmatrix} 1 & \Lambda_3 & 0 \\ -\Lambda_3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}$$

i.e.

$$\phi_1' = \phi_1 + \Lambda_3 \phi_2$$
$$\phi_2' = \phi_2 - \Lambda_3 \phi_1$$
$$\phi_3' = \phi_3$$

Compare with:

$$- \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & \Lambda_3 \\ \phi_1 & \phi_2 & \phi_3 \end{vmatrix} = \Lambda_3 \phi_2 \underline{i} - \Lambda_3 \phi_1 \underline{j}$$

So the general small angle rotation is $-\underline{\Lambda} \times \underline{\phi}$. In O(3) vector rotation:

$$\underline{\phi}' = e^{i \underline{J} \cdot \underline{\Lambda}} \phi \leftrightarrow \underline{\phi}' = \underline{\phi} - \underline{\Lambda} \times \underline{\phi}$$

Now apply the formula:

$$A_\mu' = \left(S A_\mu - \frac{i}{g} \partial_\mu S \right) S^{-1}$$

with

$$S \underline{A}_\mu = \exp\left(i \underline{J} \cdot \underline{\Lambda} \right) \underline{A}_\mu = \underline{A}_\mu - \underline{\Lambda} \times \underline{A}_\mu$$

$$\partial_\mu S = \left(i\, \partial_\mu \underline{\Lambda} \right) S$$

we obtain:

$$\underline{A}_\mu' = \left(\underline{A}_\mu - \underline{\Lambda} \times \underline{A}_\mu - \frac{i}{g} \partial_\mu \underline{\Lambda}\, S \right) S^{-1}$$

where:

$$S = e^{i \underline{J} \cdot \underline{\Lambda}} = 1 + i\, \underline{J} \cdot \underline{\Lambda} + \cdots\cdots$$

$$S^{-1} = e^{-i \underline{J} \cdot \underline{\Lambda}} = 1 - i\, \underline{J} \cdot \underline{\Lambda} + \cdots\cdots$$

so: $\underline{A}_\mu' \sim \underline{A}_\mu - \underline{\Lambda} \times \underline{A}_\mu + \frac{1}{g} \partial_\mu \underline{\Lambda} + \cdots\cdots$

This result is the Yang Mills approximation.

For Z axis rotation:

$$\Lambda_1 = \Lambda_2 = 0$$

and

$$-\underline{\Lambda} \times \underline{A}_\mu = \Lambda_3 A_{\mu 2}\, \underline{i} - \Lambda_3 A_{\mu 1}\, \underline{j}$$

$$\Rightarrow \boxed{\begin{aligned} A_{\mu 1}' &= A_{\mu 1} + \Lambda_3 A_{\mu 2} \\ A_{\mu 2}' &= A_{\mu 2} - \Lambda_3 A_{\mu 1} \\ A_{\mu 3}' &= A_{\mu 3} + \frac{1}{g} \partial_\mu \Lambda_3 \end{aligned}}$$

which is a geometrical result. In flat spacetime the third equation is $A_{\mu 3}' = A_{\mu 3}$. In general it is the same as the SU(2) result.

O(3) Matrices in General Field Theory

The field and gauge transformation equations are:

$$A'_\mu = \left(SA_\mu - \frac{i}{g}\partial_\mu S\right)S^{-1} \quad — (1)$$

$$G'_{\mu\nu} = S\, G_{\mu\nu}\, S^{-1} \quad — (2)$$

$$G_{\mu\nu} = \frac{i}{g}\left[D_\mu, D_\nu\right] \quad — (3)$$

These are the main equations of general field theory, in which A_μ is a potential, S is a rotation matrix, $G_{\mu\nu}$ the field tensor and D_μ the covariant derivative.

In the O(3) ~~equation~~ symmetry:

$$S = e^{iJ_Z\alpha}; \quad S^{-1} = e^{-iJ_Z\alpha} \quad — (4)$$

Thus:

$$S = 1 + iJ_Z\alpha - J_Z^2\frac{\alpha^2}{2!} - iJ_Z^3\frac{\alpha^3}{3!} + \dots$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \alpha\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{\alpha^2}{2!}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \dots$$

$$= \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad — (5)$$

for a rotation about the Z axis.

The inverse of (5) is found by $\alpha \rightarrow -\alpha$:

$$S^{-1} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = e^{-iJ_Z\alpha}$$

Therefore eqns (1) to (3) are operator or matrix equations.

Check

$$SS^{-1} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & -\cos\alpha\sin\alpha + \sin\alpha\cos\alpha & 0 \\ -\sin\alpha\cos\alpha + \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Calculation of $\partial_\mu S$

This term depends on $\alpha := \alpha(x^\mu)$. If α is not a function of x^μ it is zero. This means that the quaternions must be a function of x^μ also.

$$\partial_\mu S = \partial_\mu \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Use: $\dfrac{dy}{dx} = \dfrac{df}{dx}\dfrac{dy}{df}$, so if:

$$y = \cos(f(x)) \; ; \quad \frac{dy}{dx} = -f'(x)\sin(f(x)).$$

So:

$$\partial_\mu(\cos\alpha(x^\mu)) = -\partial_\mu\alpha \sin\alpha$$

$$\partial_\mu(\sin\alpha(x^\mu)) = \partial_\mu\alpha \cos\alpha$$

$$\boxed{\partial_\mu S = \partial_\mu\alpha \begin{bmatrix} -\sin\alpha & \cos\alpha & 0 \\ -\cos\alpha & -\sin\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

Note

: If $\partial_\mu \alpha = 0$, then $\partial_\mu S = 0$, corresponding to flat spacetime. Covariant derivatives characterise curved spacetime as in general relativity.

O(3) Definition of A_μ

$$\boxed{A_\mu := J^a A_\mu^a}$$

In general gauge field theory the potential four-vector is expressed as an operator, an infinitesimal rotation generator, J^a, multiplied by a double indexed scalar component denoted A_μ^a. This method originates in Yang Mills theory.

In O(3) ~~is~~, for a Z axis rotation:

$$A_\mu = \cancel{\text{[illegible]}}\ J^Z A_\mu^Z$$

In this notation, the placing of Z as an upper or lower index has no significance, whereas μ is covariant or contravariant as usual. Thus:

$$\boxed{A_\mu = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} A_\mu^Z}$$

So we are now ready to work out eqns. (1) to (3)

Start with:

$$\partial_\mu S S^{-1} = \partial_\mu \alpha \begin{bmatrix} -\sin\alpha & \cos\alpha & 0 \\ -\cos\alpha & -\sin\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \partial_\mu \alpha \begin{bmatrix} -\sin\alpha\cos\alpha + \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha & 0 \\ -\cos^2\alpha - \sin^2\alpha & \cos\alpha\sin\alpha - \sin\alpha\cos\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \partial_\mu \alpha \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{-\frac{i}{g} \partial_\mu S S^{-1} = \frac{J_z}{g} \partial_\mu \alpha}$$

Note This is the inhomogeneous term responsible for the Aharonov Bohm effects. The scalar g is a dimensionality coefficient introduced in the definition of the covariant derivative. The operator J_z is the infinitesimal rotation generator about z.

Check

$$S = e^{iJ_z\alpha}$$

$$\partial_\mu S = (i J_z \partial_\mu \alpha) S$$

$$\partial_\mu S S^{-1} = i J_z \partial_\mu \alpha$$

The Term $SA_\mu S^{-1}$

$$SA_\mu S^{-1} = -iA_\mu^z \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= -iA_\mu^z \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin\alpha & \cos\alpha & 0 \\ -\cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= -iA_\mu^z \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= A_\mu^z J_z = A_\mu$$

$$\boxed{SA_\mu S^{-1} = A_\mu}$$

Overall Result

$$A_\mu^z J_z \longrightarrow \left(A_\mu^z + \frac{1}{g} \partial_\mu \alpha \right) J_z$$

i.e. $$\boxed{A_\mu^z \longrightarrow A_\mu^z + \frac{1}{g} \partial_\mu \alpha}$$

Check

In flat spacetime, $\partial_\mu \alpha = 0$, and $A_\mu^z \rightarrow A_\mu^z$. This means that rotation about Z leaves the Z component of A_μ^z unaffected, q.e.d.

Transformation of the Field Tensor

! The field tensor in general gauge field theory is defined ~~as~~ in a Taylor series. ~~As~~ One term of this series is a commutator of covariant derivatives. The latter are defined in terms of the potential A_μ, which is Feynman's "universal influence". So field theory depends on ~~the~~ structure of spacetime, or the structure of the vacuum.

So:

$$\boxed{\begin{aligned} G_{\mu\nu} &:= \frac{i}{g}\left[D_\mu, D_\nu\right] \quad \text{———} \quad (1a) \\ G_{\mu\nu}' &= S\, G_{\mu\nu}\, S^{-1} \quad \text{———} \quad (1b) \end{aligned}}$$

Derivation of the $B^{(3)}$ Field

$$G_{\mu\nu} := \frac{i}{g}\left[\partial_\mu - igA_\mu,\ \partial_\nu - igA_\nu\right]$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right].$$

Define component matrices of A_μ in the basis ((1), (2), (3)).

$$G_{\mu\nu}^{(3)} = \partial_\mu A_\nu^{(3)} - \partial_\nu A_\mu^{(3)} - ig\left[A_\mu^{(1)}, A_\nu^{(2)}\right],$$

so:

$$G_{12}^{(3)} = -ig\left(A_1^{(1)}A_2^{(2)} - A_2^{(1)}A_1^{(2)}\right),$$

or

$$G_{XY}^{(3)} = -ig\left(A_X^{(1)}A_Y^{(2)} - A_Y^{(1)}A_X^{(2)}\right)$$

In vector form:

$$\underline{B}^{(2)*} = -ig\, \underline{A}^{(1)} \times \underline{A}^{(2)}.$$

Rotation About Z Axis

$$G^{Z\,\prime}_{XY} = S\, G^{Z}_{XY}\, S^{-1}.$$

$$\begin{bmatrix} 0 & -B_Z & 0 \\ B_Z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -B_Z & 0 \\ B_Z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -B_Z & 0 \\ B_Z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

i.e. $\boxed{B_Z \rightarrow B_Z}$

Note

The magnetic field's Z component is invariant under rotation about the Z axis. In general, the O(3) gauge transform is a rotation, which produces:

$$\boxed{\begin{array}{l} A^Z_Z \rightarrow A^Z_Z + \frac{1}{g}\partial_Z \alpha \\ B_Z \rightarrow B_Z \end{array}}$$

The characteristic inhomogeneous term arises from the fact that the Euler angle α is in general a function of spacetime, x^μ, giving rise to Aharonov Bohm effects.

Radiative Aharonov Bohm Effects

In $O(3)$ gauge theory. These are due to individual rotations of $\underline{A}^{(1)}$ and $\underline{A}^{(2)}$. Begin by considering gauge transformations of $\underline{A}^{(1)}$ and $\underline{A}^{(2)}$ when these are plane waves:

$$\underline{A}^{(1)} \rightarrow S\underline{A}^{(1)}S^{-1} - \frac{i}{g}\partial_\mu S S^{-1}$$

$$\underline{A}^{(2)} \rightarrow S\underline{A}^{(2)}S^{-1} + \frac{i}{g}\left(\partial_\mu S S^{-1}\right)^*$$

where:

$$\underline{A}^{(1)} = \frac{A^{(0)}}{\sqrt{2}}(\underline{i} - i\underline{j})e^{i\phi} \; ; \quad \underline{A}^{(2)} = \frac{A^{(0)}}{\sqrt{2}}(\underline{i} + i\underline{j})e^{-i\phi}$$

$$:= A_x^{(1)}\underline{i} + A_y^{(1)}\underline{j} \; ; \qquad := A_x^{(2)}\underline{i} + A_y^{(2)}\underline{j} .$$

i.e.

$$A_x^{(1)} = \frac{A^{(0)}}{\sqrt{2}}e^{i\phi} \; ; \quad A_y^{(1)} = -i\frac{A^{(0)}}{\sqrt{2}}e^{i\phi} \; ;$$

$$A_x^{(2)} = \frac{A^{(0)}}{\sqrt{2}}e^{-i\phi} \; ; \quad A_y^{(2)} = i\frac{A^{(0)}}{\sqrt{2}}e^{-i\phi} .$$

$$A^{(1)} = A_x^{(1)}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} + A_y^{(1)}\begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}$$

$$A^{(2)} = A_x^{(2)}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} + A_y^{(2)}\begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}$$

Rotation About Z Axis

(i) of $A_x^{(1)}$

$$SA_x^{(1)}S^{-1} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} A_x^{(1)}$$

$$= \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ i\sin\alpha & i\cos\alpha & 0 \end{bmatrix} A_x^{(1)}$$

$$= \begin{bmatrix} 0 & 0 & -i\sin\alpha \\ 0 & 0 & -i\cos\alpha \\ i\sin\alpha & i\cos\alpha & 0 \end{bmatrix} A_x^{(1)}$$

$$\therefore \boxed{\begin{aligned} SA_x^{(1)}S^{-1} &= A_x^{(1)} \begin{bmatrix} 0 & 0 & -i\sin\alpha \\ 0 & 0 & -i\cos\alpha \\ i\sin\alpha & i\cos\alpha & 0 \end{bmatrix} \\ A_x^{(1)} &= A_x^{(1)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \end{aligned}}$$

Physical Consequence

The vector $\underline{A}^{(1)}$ is changed by an $O(3)$ gauge transformation.

2) of $A_y^{(1)}$

$$SA_y^{(1)}S^{-1} = A_y^{(1)} \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= A_y^{(1)} \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i\cos\alpha & i\sin\alpha & 0 \end{bmatrix}$$

$$\boxed{\begin{aligned} SA_y^{(1)}S^{-1} &= A_y^{(1)} \begin{bmatrix} 0 & 0 & i\cos\alpha \\ 0 & 0 & -i\sin\alpha \\ -i\cos\alpha & i\sin\alpha & 0 \end{bmatrix} \\ A_y^{(1)} &= A_y^{(1)} \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} \end{aligned}}$$

Check

The commutator $[SA_x^{(1)}S^{-1}, SA_y^{(1)}S^{-1}]$ should be a z axis rotation:

$$[SA_x^{(1)}S^{-1}, SA_y^{(1)}S^{-1}] = (SA_x^{(1)}S^{-1}SA_y^{(1)}S^{-1} - SA_y^{(1)}S^{-1}SA_x^{(1)}S^{-1})$$

$$= A_x^{(1)}A_y^{(1)} \begin{bmatrix} 0 & 0 & -i\sin\alpha \\ 0 & 0 & -i\cos\alpha \\ i\sin\alpha & i\cos\alpha & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & i\cos\alpha \\ 0 & 0 & -i\sin\alpha \\ -i\cos\alpha & i\sin\alpha & 0 \end{bmatrix}$$

$$- A_x^{(1)}A_y^{(1)} \begin{bmatrix} 0 & 0 & i\cos\alpha \\ 0 & 0 & -i\sin\alpha \\ -i\cos\alpha & i\sin\alpha & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -i\sin\alpha \\ 0 & 0 & -i\cos\alpha \\ i\sin\alpha & i\cos\alpha & 0 \end{bmatrix}$$

$$= A_x^{(1)}A_y^{(1)} \left(\begin{bmatrix} -\sin\alpha\cos\alpha & \sin^2\alpha & 0 \\ -\cos^2\alpha & \cos\alpha\sin\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} -\sin\alpha\cos\alpha & -\cos^2\alpha & 0 \\ \sin^2\alpha & -\sin\alpha\cos\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)$$

$$= A_x^{(1)} A_y^{(1)} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = i A_x^{(1)} A_y^{(1)} J_Z$$

<u>Result</u>

A rotation about the Z axis changes the X and Y components of the potential, but leaves the Z component, proportional to $\underline{B}^{(3)}$, unchanged. However, the polar A_Z is changed by the same rotation to $A_Z + \partial_Z \alpha$. Therefore $\underline{A}^{(1)} \times \underline{A}^{(2)}$ must be proportional to $\underline{B}^{(3)}$.

Complex Vector Potentials

$$A^{(1)} = A_x^{(1)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} + A_y^{(1)} \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}$$

$$SA^{(1)}S^{-1} = SA_x^{(1)}S^{-1} + SA_y^{(1)}S^{-1}$$

$$= A_x^{(1)} \begin{bmatrix} 0 & 0 & -i\sin\alpha \\ 0 & 0 & -i\cos\alpha \\ i\sin\alpha & i\cos\alpha & 0 \end{bmatrix} + A_y^{(1)} \begin{bmatrix} 0 & 0 & i\cos\alpha \\ 0 & 0 & -i\sin\alpha \\ -i\cos\alpha & i\sin\alpha & 0 \end{bmatrix}$$

$$= \frac{A^{(0)}}{\sqrt{2}} e^{i\phi} \begin{bmatrix} 0 & 0 & -i\sin\alpha + \cos\alpha \\ 0 & 0 & -i\cos\alpha - \sin\alpha \\ i\sin\alpha - \cos\alpha & i\cos\alpha + \sin\alpha & 0 \end{bmatrix}$$

$$SA^{(2)}S^{-1} = SA_x^{(2)}S^{-1} + SA_y^{(2)}S^{-1}$$

$$= \frac{A^{(0)}}{\sqrt{2}} e^{-i\phi} \begin{bmatrix} 0 & 0 & -i\sin\alpha - \cos\alpha \\ 0 & 0 & -i\cos\alpha + \sin\alpha \\ i\sin\alpha + \cos\alpha & i\cos\alpha - \sin\alpha & 0 \end{bmatrix}$$

$$[SA^{(1)}S^{-1}, SA^{(2)}S^{-1}] = SA^{(1)}S^{-1}SA^{(2)}S^{-1} - SA^{(2)}S^{-1}SA^{(1)}S^{-1}$$

$$\begin{bmatrix} 0 & 0 & -i\sin\alpha+\cos\alpha \\ 0 & 0 & -i\cos\alpha-\sin\alpha \\ i\sin\alpha-\cos\alpha & i\cos\alpha+\sin\alpha & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & -i\sin\alpha-\cos\alpha \\ 0 & 0 & -i\cos\alpha+\sin\alpha \\ i\sin\alpha+\cos\alpha & i\cos\alpha-\sin\alpha & 0 \end{bmatrix}$$

$$-\begin{bmatrix} 0 & 0 & -i\sin\alpha-\cos\alpha \\ 0 & 0 & -i\cos\alpha+\sin\alpha \\ i\sin\alpha+\cos\alpha & i\cos\alpha-\sin\alpha & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & -i\sin\alpha+\cos\alpha \\ 0 & 0 & -i\cos\alpha-\sin\alpha \\ i\sin\alpha-\cos\alpha & i\cos\alpha+\sin\alpha & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sin^2\alpha+\cos^2\alpha & i(\cos^2\alpha+\sin^2\alpha) & 0 \\ -i(\sin^2\alpha+\cos^2\alpha) & \cos^2\alpha+\sin^2\alpha & 0 \\ 0 & 0 & 2(\sin^2\alpha+\cos^2\alpha) \end{bmatrix} - \begin{bmatrix} \sin^2\alpha+\cos^2\alpha & -i(\cos^2\alpha+\sin^2\alpha) & 0 \\ i(\sin^2\alpha+\cos^2\alpha) & \sin^2\alpha+\cos^2\alpha & 0 \\ 0 & 0 & 2(\sin^2\alpha+\cos^2\alpha) \end{bmatrix}$$

$$= 2\begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = -2J_Z$$

$$\boxed{[\underset{\text{polar}}{SA^{(1)}S^{-1}}, \underset{\text{polar}}{SA^{(2)}S^{-1}}] = -A^{(0)2} J_Z \propto B^{(3)}_{\text{axial}}}.$$

Result

In vector notation:

$$\boxed{\underline{B}^{(3)} = -i\frac{\kappa}{A^{(0)}}\underline{A}^{(1)} \times \underline{A}^{(2)}}$$

qed

The term $[A^{(1)}, A^{(2)}]$

$$A^{(1)} = \frac{A^{(0)}}{\sqrt{2}} e^{i\phi}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ -1 & i & 0 \end{bmatrix}; \quad A^{(2)} = \frac{A^{(0)}}{\sqrt{2}} e^{-i\phi}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -i \\ 1 & i & 0 \end{bmatrix}$$

$$[A^{(1)}, A^{(2)}] = A^{(1)}A^{(2)} - A^{(2)}A^{(1)}$$

$$= \frac{1}{2}A^{(0)2}\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ -1 & i & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -i \\ 1 & i & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -i \\ 1 & i & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ -1 & i & 0 \end{bmatrix}\right).$$

$$= \frac{1}{2} A^{(0)2} \left(\begin{bmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) = -A^{(0)2} J_z$$ (20)

$$\boxed{[A^{(1)}, A^{(2)}] = [SA^{(1)}S^{-1}, SA^{(2)}S^{-1}]}$$

Result

The gauge transform (rotation about z) changes $\underline{A}^{(1)}$ and $\underline{A}^{(2)}$ but leaves $\underline{A}^{(1)} \times \underline{A}^{(2)}$ unchanged. This is a simple geometrical result consistent with $\underline{B}^{(3)} \rightarrow \underline{B}^{(3)}$. Therefore to see the optical Aharonov Bohm effect we must investigate $\underline{A}^{(1)}$ and $\underline{A}^{(2)}$ individually. This is the simple case of Z axis rotation.

The Inhomogeneous Terms

$$A^{(1)} \rightarrow SA^{(1)}S^{-1} - \frac{i}{g}\left(\partial_\mu S S^{-1}\right)^{(1)} \quad -(1)$$

$$A^{(2)} \rightarrow SA^{(2)}S^{-1} + \frac{i}{g}\left(\partial_\mu S S^{-1}\right)^{(2)} \quad -(2)$$

Eqn. (2) is the complex conjugate of eqn. (1)

$$[A^{(1)}, A^{(2)}] = [SA^{(1)}S^{-1}, SA^{(2)}S^{-1}] \quad -(3)$$

for a Z axis rotation.

We have:

$$-\frac{i}{g}\left(\partial_\mu S S^{-1}\right)^{(1)} = \frac{1}{g}\partial_\mu \alpha \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{J_z}{g}\partial_\mu \alpha$$

$$\text{c.c.} = \frac{1}{g}\partial_\mu \alpha \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = -\frac{J_z}{g}\partial_\mu \alpha$$

Commutator of Inhomogeneous Terms

$$= \frac{1}{g^2}(\partial_\mu \alpha)(\partial_\mu \alpha)^* \left(\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)$$

$$= 0$$

Cross Terms

The sum of cross terms must be zero $\because$ eqns (1) to (3):

$$\Rightarrow (SA^{(1)}S^{-1})(\partial_\mu S S^{-1})^{(2)} - (\partial_\mu S S^{-1})^{(1)}(SA^{(2)}S^{-1})$$
$$+ (SA^{(2)}S^{-1})(\partial_\mu S S^{-1})^{(1)} - (\partial_\mu S S^{-1})^{(2)}(SA^{(1)}S^{-1}) = 0$$

Individual Terms

$$(SA^{(1)}S^{-1})(\partial_\mu S S^{-1})^{(2)} - (\partial_\mu S S^{-1})^{(2)}(SA^{(1)}S^{-1})$$

$$= \partial_\mu \alpha \frac{A^{(0)}}{\sqrt{2}} e^{i\phi} \left(\begin{bmatrix} 0 & 0 & -i\sin\alpha + \cos\alpha \\ 0 & 0 & -i\cos\alpha - \sin\alpha \\ i\sin\alpha - \cos\alpha & i\cos\alpha + \sin\alpha & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -i\sin\alpha + \cos\alpha \\ 0 & 0 & -i\cos\alpha - \sin\alpha \\ i\sin\alpha - \cos\alpha & i\cos\alpha + \sin\alpha & 0 \end{bmatrix} \right)$$

$$= \partial_\mu \alpha \frac{A^{(0)}}{\sqrt{2}} e^{i\phi} \begin{bmatrix} 0 & 0 & i\cos\alpha + \sin\alpha \\ 0 & 0 & -i\sin\alpha + \cos\alpha \\ -i\cos\alpha - \sin\alpha & i\sin\alpha - \cos\alpha & 0 \end{bmatrix} \quad \text{— (1)}$$

$$(SA^{(2)}S^{-1})(\partial_\mu S S^{-1})^{(1)} - (\partial_\mu S S^{-1})^{(1)}(SA^{(2)}S^{-1})$$

$$= \partial_\mu \alpha \frac{A^{(0)}}{\sqrt{2}} e^{-i\phi} \left(\begin{bmatrix} 0 & 0 & -i\sin\alpha - \cos\alpha \\ 0 & 0 & -i\cos\alpha + \sin\alpha \\ i\sin\alpha + \cos\alpha & i\cos\alpha - \sin\alpha & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -i\sin\alpha - \cos\alpha \\ 0 & 0 & -i\cos\alpha + \sin\alpha \\ i\sin\alpha + \cos\alpha & i\cos\alpha - \sin\alpha & 0 \end{bmatrix} \right)$$

$$= \partial_\mu \alpha \frac{A^{(0)}}{\sqrt{2}} e^{-i\phi} \begin{bmatrix} 0 & 0 & i\cos\alpha - \sin\alpha \\ 0 & 0 & -i\sin\alpha - \cos\alpha \\ -i\cos\alpha + \sin\alpha & i\sin\alpha + \cos\alpha & 0 \end{bmatrix} \quad \text{— (2)}$$

Add (1) and (2) to obtain:

$$\partial_\mu \alpha \frac{A^{(0)}}{\sqrt{2}} \left(e^{i\phi} \begin{bmatrix} 0 & 0 & i\cos\alpha + \sin\alpha \\ 0 & 0 & -i\sin\alpha + \cos\alpha \\ -i\cos\alpha - \sin\alpha & i\sin\alpha - \cos\alpha & 0 \end{bmatrix} + e^{-i\phi} \begin{bmatrix} 0 & 0 & i\cos\alpha - \sin\alpha \\ 0 & 0 & -i\sin\alpha - \cos\alpha \\ -i\cos\alpha + \sin\alpha & i\sin\alpha + \cos\alpha & 0 \end{bmatrix} \right) = 0$$

$$\partial_\mu \alpha \frac{A^{(0)}}{\sqrt{2}} \left(e^{i\phi} \begin{bmatrix} 0 & 0 & ie^{-i\alpha} \\ 0 & 0 & e^{-i\alpha} \\ -ie^{-i\alpha} & -e^{-i\alpha} & 0 \end{bmatrix} + e^{-i\phi} \begin{bmatrix} 0 & 0 & +ie^{i\alpha} \\ 0 & 0 & -e^{i\alpha} \\ -ie^{+i\alpha} & e^{i\alpha} & 0 \end{bmatrix} \right) = 0$$

<u>Compare Terms</u>:

$$e^{i(\phi-\alpha)} = -e^{-i(\phi-\alpha)} \quad \text{— (3)}$$

$$\cos(\phi - \alpha) = 0$$

$$\boxed{\cancel{\alpha = \phi \mp 2\pi n}}$$

$$\phi - \alpha = \pm \frac{\pi}{2} = \pm(2n+1)\frac{\pi}{2}$$

— (4)

$$\therefore \partial_Z \alpha = -\kappa,$$

κ = wavenumber magnitude.

$$\boxed{\begin{aligned} A_Z &\longrightarrow A_Z - \frac{\kappa}{g} = A_Z - A^{(0)} \\ \underline{B}^{(3)} &\longrightarrow \underline{B}^{(3)}; \quad \underline{A}^{(1)} \times \underline{A}^{(2)} \rightarrow \underline{A}^{(1)} \times \underline{A}^{(2)} \end{aligned}}$$

SELF - CONSISTENT

Overall Conclusion

In O(3) gauge theory, there is a gauge transformed radiated Aharonov Bohm effect:

$$\boxed{\begin{aligned} &A_Z\,(\text{polar}) \rightarrow A_Z\,(\text{polar}) - A^{(0)} \\ &\underline{B}^{(3)} \rightarrow \underline{B}^{(3)} \\ &A^{(1)} \rightarrow SA^{(1)}S^{-1} - \frac{1}{g}\partial_\mu S S^{-1} \\ &\underline{A}^{(1)} \times \underline{A}^{(2)} \rightarrow \underline{A}^{(1)} \times \underline{A}^{(2)} \\ &\alpha = \phi \mp \cancel{4}2\pi n/2 \end{aligned}}$$

The challenge is to detect this experimentally.