190(4) : Orbital Equation for the Complete m Function

The starting equation is again:

$$\frac{1}{2} m \left(\frac{dr}{d\tau}\right)^2 = \frac{1}{2}\left(\frac{E^2}{mc^2} - m(r)\left(mc^2 + \frac{L^2}{mr^2}\right)\right) \quad -(1)$$

where $$m(r) = 2 - \exp\left(2\exp\left(-\frac{r}{R}\right)\right) \quad -(2)$$

Let $$m(r) = 1 + f(r) \quad -(3)$$

where $$f(r) = 1 - \exp\left(2\exp\left(-\frac{r}{R}\right)\right) \quad -(4)$$

Then:

$$\frac{1}{2}\left(\frac{E^2}{mc^2} - (1+f(r))mc^2\right) = \frac{1}{2} m \left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}(1+f(r))\frac{L^2}{mr^2} \quad -(5)$$

i.e.

$$\frac{1}{2}\left(\frac{E^2}{mc^2} - mc^2\right) = \frac{1}{2} m \left(\frac{dr}{d\tau}\right)^2 + \frac{L^2}{2mr^2} + V_a \quad -(6)$$

where the attractive potential is:

$$V_a = \frac{f(r)}{2}\left(mc^2 + \frac{L^2}{mr^2}\right) \quad -(7)$$

$$= \frac{1}{2}\left(1 - \exp\left(2\exp\left(-\frac{r}{R}\right)\right)\right)\left(mc^2 + \frac{L^2}{mr^2}\right) \quad -(8)$$

The force of attraction is:

2)

$$F = -\frac{\partial V_a}{\partial r} \quad -(9)$$

$$\boxed{F(r) = \frac{1}{2}\frac{\partial}{\partial r}\left(\exp\left(2\exp\left(-\frac{r}{R}\right)\right)\left(mc^2 + \frac{L^2}{mr^2}\right)\right)} \quad -(10)$$

and the orbital equation is:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{mr^2}{L^2}F(r) \quad -(11)$$

The orbit is found numerically from eqs. (10) and (11).

Polynomial Approximation

Eq (1) becomes:

$$\frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 = \frac{1}{2}\left(\frac{E^2}{mc^2} - \left(1 - \frac{r_0}{r} - \frac{a}{r^2} - \frac{b}{r^3} - \ldots\right)\left(mc^2 + \frac{L^2}{mr^2}\right)\right) \quad -(12)$$

$$= \frac{1}{2}\left(\frac{E^2}{mc^2} - mc^2 - \frac{L^2}{mr^2} + \left(\frac{r_0}{r} + \frac{a}{r^2} + \frac{b}{r^3} + \ldots\right)\left(mc^2 + \frac{L^2}{mr^2}\right)\right) \quad -(13)$$

i.e.

$$\frac{1}{2}\left(\frac{E^2}{mc^2} - mc^2\right) = \frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 + \frac{L^2}{2mr^2} + V_a \quad -(14)$$

where:

$$V_a = -\frac{1}{2}\left(\frac{r_0}{r} + \frac{a}{r^2} + \frac{b}{r^3} + \ldots\right)\left(mc^2 + \frac{L^2}{mr^2}\right) \quad -(15)$$

is the potential of attraction. Here:

$$r_0 = 2MG / c^2 \quad -(16)$$

The force of attraction between m and M is:

$$F(r) = -\partial V_a / \partial r \quad -(17)$$

and

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{mr^2}{L^2} F(r) \quad -(18)$$

For given a and b, the orbit is found from eqs. (15) to (18).

Fitting Procedure:

$$m(r) = 2 - \exp\left(2\exp\left(-\frac{r}{R}\right)\right) = 1 - \frac{r_0}{r} - \frac{a}{r^2} - \frac{b}{r^3} - \ldots \quad -(19)$$

The rigorous force law is eq. (10), but eq. (19) fits it to the "traditional" format.