1)

190 (3) : Derivation of Orbits from a Function.

In a spherical spacetime the equation of motion is:

$$\frac{1}{2} m \left(\frac{dr}{d\tau}\right)^2 = \frac{1}{2}\left(\frac{E^2}{mc^2} - m(r)\left(mc^2 + \frac{L^2}{mr^2}\right)\right) \quad -(1)$$

where $m(r)$ is assumed in the first instance to be a function only of r and R. Here τ is the proper time, E is the total energy and L the angular momentum. E and L are constants of motion. In eqn. (1), r is the radial coordinate, and m is the mass of the attracted object.

In the "Schwarzschild" approximation:

$$m(r) \sim 1 - \frac{r_0}{r} \quad -(2)$$

Here:

$$r_0 = 2MG/c^2. \quad -(3)$$

Here M is the mass of the sun, G is Newton's constant and c is the speed of light.

In higher order approximations:

$$m(r) \sim 1 - \frac{r_0}{r} - \frac{a}{r^2} - \frac{b}{r^3} - \cdots \quad -(4)$$

For ease of calculation an effective potential is obtained from eq. (1) and is used to obtain the way in which r depends on the angle θ of the planar orbit. In general:

$$m(r) = \exp\left(2\exp\left(-\frac{r}{R}\right)\right) \quad -(5)$$

here R is a constant.

2) From eqs. (1) to (3):

$$\frac{1}{2}\left(\frac{E^2}{mc^2} - mc^2\right) = \frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 + V \qquad -(6)$$

which is the relativistic generalization of:

$$E = T + V \qquad -(7)$$

where T and V are the kinetic and potential energies.

Here:

$$V = -\frac{mMG}{r} - \frac{MGL^2}{mc^2r^3} + \frac{L^2}{2mr^2} \qquad -(8)$$

Although force and potential energy are replaced in general relativity by the infinitesimal line element, the effective potential is used as in eqn. (8).

The effective force is:

$$F = -\frac{\partial V}{\partial r} = -\frac{mMG}{r^2} - \frac{3MGL^2}{mc^2r^4} + \frac{L^2}{mr^3} \qquad -(9)$$

In the received opinion the first term is the Hooke/Newton inverse square law, the second term is the relativistic attraction and the third term is the centripetal repulsion.

The next stage of the calculation is to use the classical equation:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{mr^2}{L^2}F(r) \qquad -(10)$$

3) To calculate the orbit. Use:

$$u = \frac{1}{r} \qquad - (11)$$

and

$$\frac{d^2 u}{d\theta^2} + u = - \frac{m}{L^2 u^2} F(u) \qquad - (12)$$

In eqs. (10) and (12) the attractive part of the force is used:

$$F(r) = - \frac{mMG}{r} - \frac{mGL^2}{mc^2 r^3} \qquad - (13)$$

so:

$$\boxed{\frac{d^2 u}{d\theta^2} + u = \frac{Gm^2 M}{L^2} + \frac{3GM}{c^2} u^2} \qquad - (14)$$

Computer algebra should be used at this stage to show that eq. (14) is the equation of the precessing ellipse.

The analytical solution of eq. (14) is very cumbersome but is given here for completeness. First define:

$$\frac{1}{d} = \frac{Gm^2 M}{L^2} \quad , \quad \delta = \frac{3GM}{c^2} \qquad - (15)$$

so:

$$\boxed{\frac{d^2 u}{d\theta^2} + u = \frac{1}{d} + \delta u^2} \qquad - (16)$$

Maxima and Mathematica may be able to

4)

, solve and graph this directly. The trial solution in the analytical method is :

$$\delta \sim 0 \quad - (17)$$

So

$$u_1 = \frac{1}{d}\left(1 + \epsilon \cos\theta\right) \quad - (18)$$

which is a static ellipse. This solution gives Newtonian orbits and Kepler's laws.

The angle θ is measured from the perihelion, where u_1 is maximum and r_1 the minimum at :

$$\theta = 0. \quad - (19)$$

From eqs. (16) and (18) :

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{d} + \frac{\delta}{d^2}\left(1 + 2\epsilon\cos\theta + \epsilon^2\cos^2\theta\right)$$

$$= \frac{1}{d} + \frac{\delta}{d^2}\left(1 + 2\epsilon\cos\theta + \frac{\epsilon^2}{2}\left(1 + \cos 2\theta\right)\right)$$

$$\quad - (20)$$

A second trial function is added to u_1 to reproduce the RHS of eq. (20). By inspection, this is :

$$u_p = \frac{\delta}{d^2}\left(\left(1 + \frac{\epsilon^2}{2}\right) + \epsilon\theta\sin\theta - \frac{\epsilon^2}{6}\cos 2\theta\right)$$

$$\quad - (21)$$

will

$$u_2 = u_1 + u_p \quad - (22)$$

In this approximation the solution of eq. (16) is

$$u = u_2 \quad - (23)$$

$$! = \left(\frac{1}{d}(1 + \epsilon\cos\theta) + \frac{\delta\epsilon}{d^2}\theta\sin\theta\right) + \left(\frac{\delta}{d^2}\left(1 + \frac{\epsilon^2}{2}\right) - \frac{\delta\epsilon^2}{6d^2}\cos 2\theta\right)$$

Considering the terms in the second bracket, the received equation asserts that the first is a constant and the second a small periodic motion, a disturbance of the Keplerian motion. So these terms are neglected. However, in the case of satellites this disturbance should be observable. It would show up with Maxima and Mathematica. In the received equation it is considered that neither term contributes to any change in the position of the apsides.

In the received equation the first term is denoted the secular term:

$$u_{secular} = \frac{1}{d}\left(1 + \epsilon\cos\theta + \frac{\delta\epsilon}{d}\theta\sin\theta\right) \qquad -(24)$$

which is approximately, within $1/d$:

$$1 + \epsilon\cos\left(\theta - \frac{\delta}{d}\theta\right) = 1 + \epsilon\left(\cos\theta\cos\frac{\delta\theta}{d} + \sin\theta\sin\frac{\delta\theta}{d}\right) \qquad -(25)$$

$$\sim 1 + \epsilon\cos\theta + \frac{\delta\epsilon}{d}\theta\sin\theta \qquad -(26)$$

because: $\delta << d \qquad -(27)$

so: $\cos\frac{\delta\theta}{d} \sim 1, \qquad \sin\frac{\delta\theta}{d} \sim \frac{\delta\theta}{d}$

∴ So:

$$u_{secular} \sim \frac{1}{\alpha}\left(1 + \epsilon \cos\left(\theta - \frac{\delta\theta}{\alpha}\right)\right) \quad -(28)$$

The position of the perihelion is defined by:

$$t = 0, \quad \theta = 0, \quad -(29)$$

i.e.

$$u_{secular} = \frac{1}{\alpha}\left(1 + \epsilon\right) \quad -(30)$$

This value is obtained when:

$$\theta - \frac{\delta\theta}{\alpha} = 2\pi n, \quad -(31)$$

$$n = 0, 1, 2, \ldots \quad -(32)$$

For one revolution of the orbit:

$$n = 1. \quad -(33)$$

In this case:

$$\theta = 2\pi\left(1 - \frac{\delta}{\alpha}\right)^{-1} \sim 2\pi\left(1 + \frac{\delta}{\alpha}\right) \quad -(34)$$

∴ the perihelion is displaced by:

$$\boxed{\Delta \sim 2\pi\frac{\delta}{\alpha} = 6\pi\left(\frac{GmM}{cL}\right)^2}$$

$$-(35)$$

This is known as the procession of the perihelion.

is worked out in astronomy by defining:

1)

$$a = \frac{d}{1 - \epsilon^2} \quad - (36)$$

here ϵ is the eccentricity of the orbit. The quantity d is:

$$d = \frac{L^2}{mk} \quad - (37)$$

where

$$k = mMG \quad - (38)$$

So.

$$\boxed{\Delta = \frac{6\pi GM}{ac^2(1 - \epsilon^2)} \quad - (39)}$$

Planet	Precessional Rate (seconds of arc / century)	
	Calculated	Observed
Mercury	43.03	43.11 ± 0.45
Venus	8.63	8.4 ± 4.8
Earth	3.84	5.0 ± 1.2
Mars	1.35	—
Jupiter	0.06	—

In fact the agreement and precision given by Marion and Thornton is this table is not convincing.