1) 190(9): Summary of Metrical Method of Deriving Orbits

Consider an orbit in the plane defined by:

$$dZ = 0. \quad -(1)$$

In the spherically symmetric spacetime the infinitesimal line element is:

$$ds^2 = c^2 d\tau^2 = m c^2 dt^2 - m^{-1} dr^2 - r^2 d\theta^2 \quad -(2)$$

in cylindrical polar coordinates. The hamiltonian is:

$$H = \frac{1}{2} mc^2 = \frac{1}{2} m \left(m(r) \left(\frac{dt}{d\tau}\right)^2 - \frac{1}{m(r)} \left(\frac{dr}{d\tau}\right)^2 - r^2 \left(\frac{d\theta}{d\tau}\right)^2 \right) -(3)$$

The constants of motion are the total energy:

$$E = m(r) mc^2 \frac{dt}{d\tau}, \quad -(4)$$

the angular momentum:

$$L = mr^2 \frac{d\theta}{d\tau}, \quad -(5)$$

and the linear momentum:

$$p = \frac{m}{m(r)} \frac{dr}{d\tau} \quad -(6)$$

Therefore eq. (3) is:

$$\frac{1}{2} m \left(\frac{dr}{d\tau}\right)^2 = \frac{1}{2} \left(\frac{E^2}{mc^2} - m(r) \left(mc^2 + \frac{L^2}{mr^2} \right) \right) -(7)$$

The orbital equation is obtained using:

$$\frac{dr}{d\tau} = \frac{dr}{d\theta} \frac{d\theta}{d\tau} = \frac{L}{mr^2} \frac{dr}{d\theta} \quad -(8)$$

From eqs. (7) and (8):

$$\frac{1}{m} \left(\frac{L}{r^2}\right)^2 \left(\frac{dr}{d\theta}\right)^2 = \frac{E^2}{mc^2} - m(r) \left(mc^2 + \frac{L^2}{mr^2} \right) -(9)$$

2) i.e.

$$\left(\frac{dr}{d\theta}\right)^2 = m\left(\frac{r^2}{L}\right)^2\left(\frac{E^2}{mc^2} - m(r)\left(mc^2 + \frac{L^2}{mr^2}\right)\right)$$

$$= r^4\left(\frac{1}{b^2} - m(r)\left(\frac{1}{a^2} + \frac{1}{r^2}\right)\right) \quad -(10)$$

where

$$a = \frac{L}{mc}, \quad b = \frac{Lc}{E} \quad -(11)$$

are constants of motion. They have units of length. So

$$\boxed{\frac{dr}{d\theta} = r^2\left(\frac{1}{b^2} - m(r)\left(\frac{1}{a^2} + \frac{1}{r^2}\right)\right)^{1/2}} \quad -(12)$$

This is true for any spherically symmetric spacetime.

The tangential velocity of the particle of mass m is:

$$v = \frac{dr}{dt} = \frac{dr}{d\tau}\frac{d\tau}{dt} = \frac{dr}{d\theta}\frac{d\theta}{d\tau}\frac{d\tau}{dt} \quad -(13)$$

with

$$\frac{d\theta}{d\tau} = \frac{L}{mr^2} \quad -(14)$$

$$\frac{d\tau}{dt} = \frac{m(r)\,mc^2}{E} \quad -(15)$$

From eqs. (12) to (15):

$$\boxed{v = cb\,m(r)\left(\frac{1}{b^2} - m(r)\left(\frac{1}{a^2} + \frac{1}{r^2}\right)\right)^{1/2}} \quad -(16)$$

The angular velocity of m is:

$$\boxed{\omega = \frac{d\theta}{dt} = \frac{d\theta}{dr}\frac{dr}{dt} = c\,b\,\frac{m(r)}{r^2}} \quad -(17)$$

from eqs. (12) and (16)

The areal velocity of the orbit is:

$$\boxed{\begin{aligned}\frac{dA}{dt} &= \frac{1}{2} r^2 \omega = c\,b\,m(r) \\ &= \frac{1}{2} c^2 \frac{L}{E} m(r)\end{aligned}} \quad -(18)$$

which is the generalization of Kepler's second law.

From UFT 186 ff:

$$\boxed{m(r) = 2 - \exp\left(2\exp\left(-\frac{r}{R}\right)\right)} \quad -(19)$$

Standard Model

This uses:

$$m(r) = 1 - \frac{r_0}{r}, \quad -(20)$$

$$r_0 = \frac{2MG}{c^2}. \quad -(21)$$

1) The correct way of calculating orbits is to use $m(r)$ in eq. (12).

2) The correct second law of Kepler is found by using eq. (18) for a given $m(r)$.

3) In theory a and b can be any constants, provided they are defined by eq. (11).

4)

Newtonian Limit

The standard way of obtaining the elliptical orbit and Newtonian limit is to use eq. (7) as follows:

$$\frac{1}{2}\left(\frac{E^2}{mc^2} - mc^2\right) = \frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 - \frac{1}{2}\frac{r_0}{r}mc^2 + \frac{1}{2}\frac{L^2}{mr^2} - \frac{L^2}{2mr^3} \quad -(22)$$

$$= \frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 + V_{eff} \quad -(23)$$

where

$$V_{eff} = -\frac{mMG}{r} + \frac{L^2}{2mr^2} - \frac{L^2}{2mr^3} \quad -(24)$$

In the Newtonian limit the last term in eq. (24) is neglected and so:

$$F_{eff}\,(\text{attractive}) = -\frac{\partial V_{eff}}{\partial r}\,(\text{attractive})$$

$$= -\frac{mMG}{r^2} \quad -(25)$$

This gives the equation:

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{\alpha} \quad -(26)$$

where

$$u = \frac{1}{r}, \quad \alpha = \frac{L^2}{Gm^2M} \quad -(27)$$

which is an ellipse.

The relativistic correction (last term RHS of

i) eq. (24), gives:

$$\frac{d^2 u}{d\theta^2} + u = \frac{1}{\alpha} + \delta u^2 \quad -(28)$$

where
$$\delta = \frac{3GM}{c^2} \quad -(29)$$

<u>Criticisms</u>

1) Eq. (22) is the same as eq. (12), but eq. (12) does not give an ellipse or a precessing ellipse for:

$$n(r) = 1 - \frac{r_0}{r} \quad -(30)$$

2) Eq (22) is relativistic but eq. (26) is classical

<u>Conclusion</u>

1) The standard model does not work, eq. (12) gives a spiral for eq. (30).

2) The $n(r)$ function for a precessing ellipse must be found from eq. (12) and:

$$\frac{\alpha}{r} = 1 + \epsilon \cos(y\theta) \quad -(31)$$

When the $n(r)$ function is much more complicated than eq. (30).