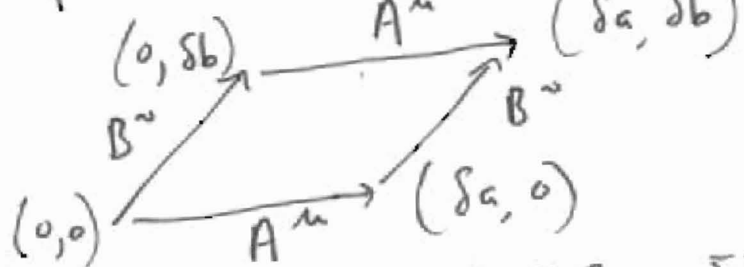
1) 190(11)

SIMPLE SUMMARY OF ANTISYMMETRY AND METRIC COMPATIBILITY.

The basic idea is that parallel transport of a vector around a closed loop will change the vector.



$$\delta V^\rho = (\delta a)(\delta b)\, A^\nu B^\mu R^\rho{}_{\sigma\mu\nu} V^\sigma \quad - (1)$$

The tensor $R^\rho{}_{\sigma\mu\nu}$ must be antisymmetric by definition.

Eq (1) is equivalent to:

$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda_{\mu\nu} D_\lambda V^\rho \quad - (2)$$

$$= -\left(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}\right) D_\lambda V^\rho + R^\rho{}_{\sigma\mu\nu} V^\sigma$$

so by definition: $\Gamma^\lambda_{\mu\nu} = -\Gamma^\lambda_{\nu\mu}$ — (3)

otherwise neither the curvature nor the torsion could be defined.

The torsion is:

$$T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \quad - (4)$$

$$= \Gamma^\lambda_{\mu\nu} - \left(-\Gamma^\lambda_{\mu\nu}\right)$$

Under transformation:

$$\Gamma^{\lambda'}_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial x^{\lambda'}}{\partial x^\lambda} \Gamma^\lambda_{\mu\nu} - \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial^2 x^{\lambda'}}{\partial x^\mu \partial x^\nu}$$

$$\Gamma^{\lambda'}_{\nu'\mu'} = \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\lambda'}}{\partial x^\lambda} \Gamma^\lambda_{\nu\mu} - \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^{\lambda'}}{\partial x^\nu \partial x^\mu}$$

2)

So:

$$\Gamma^{\lambda'}_{\mu'\nu'} - \Gamma^{\lambda'}_{\nu'\mu'}$$

$$\rightarrow \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \frac{\partial x^{\lambda'}}{\partial x^{\lambda}} \Gamma^{\lambda}_{\mu\nu} - \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \frac{\partial x^{\lambda'}}{\partial x^{\lambda}} \Gamma^{\lambda}_{\nu\mu}$$

$$= \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \frac{\partial x^{\lambda'}}{\partial x^{\lambda}} \left(\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} \right) \quad -(5)$$

So the torsion transforms as a tensor. The fact that eq. (3) is true does not change any of eqs. (4) and (5)

I suggest that the following be adapted for metric compatibility.

$$D_{\rho}\, g_{\mu\nu} = \partial_{\rho} g_{\mu\nu} - \Gamma^{\lambda}_{\rho\mu} g_{\lambda\nu} - \Gamma^{\lambda}_{\rho\nu} g_{\mu\lambda} = 0 \quad -(6)$$

$$g_{\mu\nu} = \begin{bmatrix} m(r) & 0 & 0 & 0 \\ 0 & m^{-1}(r) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad -(7)$$

The only relevant component is:

$$g_{00} = 1 / g_{11} \quad -(8)$$

So

$$\partial_{\rho} g_{00} - \Gamma^{0}_{\rho 0} g_{00} - \Gamma^{0}_{\rho 0} g_{00} = 0 \quad -(9)$$

Assume

$$m = f(r) \text{ only} \quad -(10)$$

then

$$\boxed{\partial_1 g_{00} - 2 \Gamma^{0}_{10} g_{00} = 0} \quad -(11)$$