190(6): The $m(r)$ Function for Various Orbits: Ellipse

The equation of orbits may be written as:

$$\left(\frac{dr}{d\theta}\right)^2 = r^4\left(\frac{1}{b^2} - m(r)\left(\frac{1}{a^2}+\frac{1}{r^2}\right)\right) \quad -(1)$$

Ellipse

In this case:

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(2)$$

$$\frac{dr}{d\theta} = \frac{\alpha\epsilon\sin\theta}{(1+\epsilon\cos\theta)^2} \quad -(3)$$

so

$$r^4\left(\frac{1}{b^2} - m(r)\left(\frac{1}{a^2}+\frac{1}{r^2}\right)\right) = \alpha^2\epsilon^2\left[\frac{1-\cos^2\theta}{(1+\epsilon\cos\theta)^4}\right] \quad -(4)$$

where

$$\cos\theta = \frac{1}{\epsilon}\left(\frac{\alpha}{r}-1\right) \quad -(5)$$

so:

$$\boxed{m(r) = \frac{1}{r^4}\left(\frac{1}{a^2}+\frac{1}{r^2}\right)^{-1}\left[\frac{r^4}{b^2} - \alpha^2\epsilon^2\left(\frac{1-\cos^2\theta}{(1+\epsilon\cos\theta)^4}\right)\right]} \quad -(6)$$

The function $m(r)$ can be graphed for various a, b and ϵ. The function α is a constant

2) defined by:

$$\alpha = \frac{L^2}{m^2 M G} \quad -(7)$$

where L is the angular momentum of the orbit and where ϵ is the eccentricity. Here m is the mass of the attracted object and M is the mass of the sun. Here G is Newton's constant.

The constants a and b are defined by:

$$a = \frac{L}{mc}, \quad b = \frac{cL}{E} \quad -(8)$$

where E is the total energy, a constant of motion. The angular momentum L is also a constant of motion. The eccentricity is defined by:

$$\epsilon = \left(1 + \frac{2EL^2}{m^3 M^2 G^2} \right)^{1/2} . \quad -(9)$$

The angular momentum is found from measuring α experimentally:

$$\alpha = \beta \left(1 - \epsilon^2 \right) \quad -(10)$$

where 2β is the major axis of the ellipse:

$$\beta = \frac{mMG}{2|E|} \quad -(11).$$

So

$$\boxed{|E| = \frac{mMG}{2\beta}} \quad -(12)$$

3)

$$\boxed{L = \left(m^2 M G \beta \left(1-\epsilon^2\right)\right)^{1/2}} \quad -(13)$$

So a, b, ϵ and α can be found from astronomy, the orbit being well approximated by a static ellipse. So $m(r)$ can be found for any orbit.

If the orbit is approximately circular:

$$\epsilon = 0 \quad -(14)$$

so

$$m(r) = \frac{1}{b^2}\left(\frac{1}{a^2}+\frac{1}{r^2}\right)^{-1} \quad -(15).$$

$$m(r) = \frac{a^2 r^2}{b^2\left(a^2+r^2\right)}$$

$$\boxed{m(r) = \left(\frac{E}{mc^2}\right)^2 \left(\frac{r^2}{d^2+r^2}\right)} \quad -(16)$$

in a circular orbit.

Therefore:

$$m(r) = \left(\frac{E}{mc^2}\right)^2 \left(\frac{1}{1+\left(\frac{d}{r}\right)^2}\right) \quad -(17)$$

In the limit:

4)

$$r \longrightarrow \infty \quad -(18)$$

$$n(r) \rightarrow \left(\frac{E.}{mc^2}\right)^2 \quad -(19)$$

It is known that the elliptical orbit is the result of Newtonian dynamics, i.e. of:

$$v << c \quad -(20)$$

or

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \rightarrow 1 \quad -(21)$$

of the Minkowski spacetime, where:

$$E = \gamma mc^2. \quad -(22)$$

So:

$$n(r) \rightarrow 1 \quad -(23)$$

in eq. (17).

General Computation

1) Measure the major axis of the ellipse, 2β, to find $|E|$.

2) Measure the minor axis of the ellipse, 2γ, to find L from:

$$L = (2m|E|)^{1/2}\,\gamma \quad -(24)$$

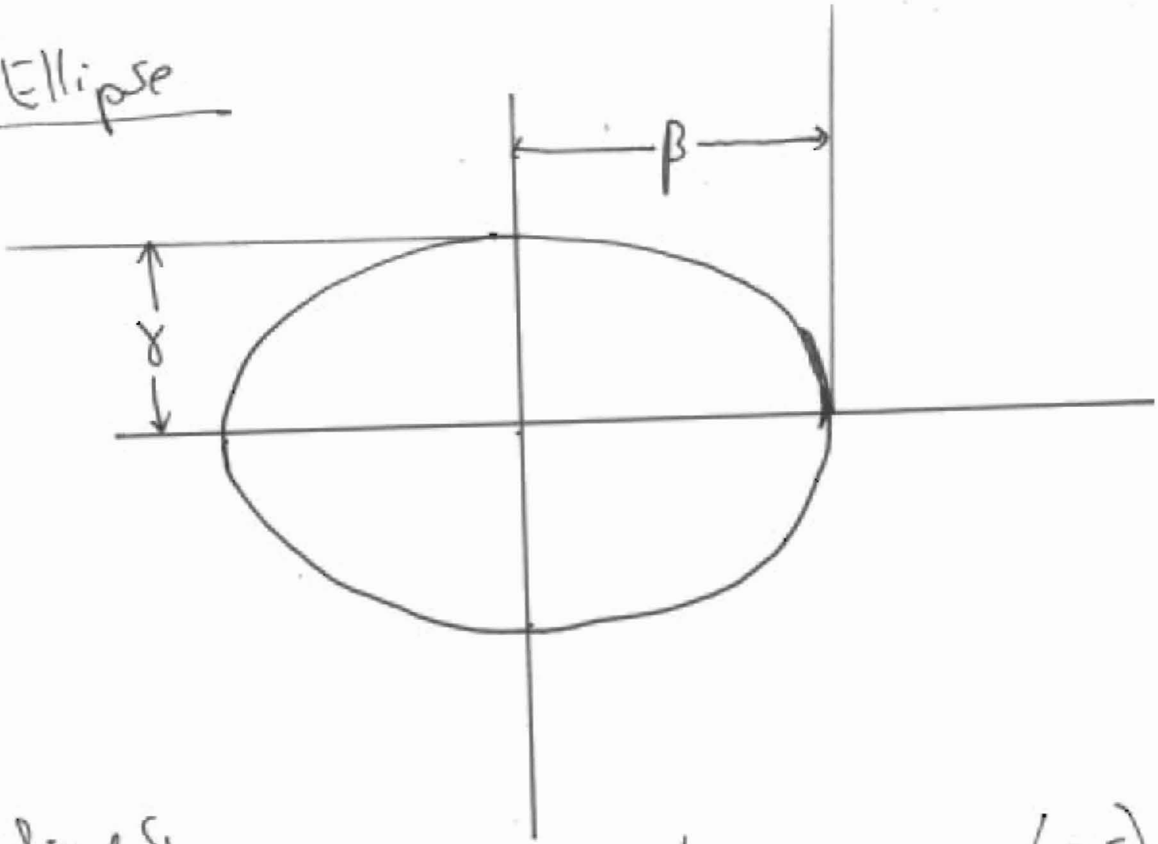
3) Find a and b from eq. (8).

4) Find α from eq. (7)

5) 5) Find ϵ from eq. (9).

6) Find $m(r)$ from eq. (6).

The Ellipse



Defining

$$\beta = \frac{\alpha}{1-\epsilon^2} \quad -(25)$$

$$\gamma = \frac{\alpha}{(1-\epsilon^2)^{1/2}} \quad -(26)$$

i.e.

$$1-\epsilon^2 = \left(\frac{\beta}{\gamma}\right)^2 \quad -(27)$$

$$\boxed{\begin{aligned} \epsilon^2 &= 1-\left(\frac{\beta}{\gamma}\right)^2 \quad -(28) \\ \alpha &= \beta(1-\epsilon^2) = \frac{\beta^3}{\gamma^2} \quad -(29) \end{aligned}}$$