1)

190(13): Derivation of the Ellipse in Newtonian and Lagrangian Theory.

In Newtonian theory:

$$E = \frac{1}{2} m v^2 - \frac{k}{r} + \frac{L^2}{2mr^2} \quad -(1)$$

where m is the mass of a particle orbiting a mass M, v is the orbital linear velocity, L is the constant angular momentum, r is the distance between m and M and:

$$k = mMG \quad -(2)$$

here G is Newton's constant. So:

$$v = \frac{dr}{dt} = \left(\frac{2}{m} \left(E + \frac{k}{r} \right) - \frac{L^2}{m^2 r^2} \right)^{1/2} \quad -(3)$$

Let

$$U = -\frac{k}{r} = -\frac{mMG}{r} \quad -(4)$$

so

$$F = -\frac{\partial U}{\partial r} = -\frac{mMG}{r^2} \quad -(5)$$

the inverse square law of Robert Hooke.

The angular momentum is:

$$L = mr^2 \omega = mr^2 \frac{d\theta}{dt} \quad -(6)$$

so

$$\frac{d\theta}{dt} = \frac{L}{mr^2} \quad -(7)$$

The areal velocity is:

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{L}{2m} = \text{constant}. \quad -(8)$$

This is Kepler's second law (1609).

The orbit is given by:

$$\frac{d\theta}{dr} = \frac{d\theta}{dt}\frac{dt}{dr} \quad -(9)$$

$$= \frac{L}{r^2}\left(2m\left(E - U - \frac{L^2}{2mr^2}\right)\right)^{-1/2}$$

This gives the ellipse by integration:

$$\theta = \int \frac{L}{r^2}\left(2m\left(E + \frac{k}{r} - \frac{L^2}{2mr^2}\right)\right)^{-1/2} dr$$

$$-(10)$$

The ellipse is:

$$\frac{\alpha}{r} = 1 + \epsilon \cos\theta \quad -(11)$$

$$\alpha = \frac{L^2}{mk}, \quad \epsilon = \left(1 + \frac{2EL^2}{mk^2}\right)^{1/2} \quad -(12)$$

The Lagrangian is:

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r) \quad -(13)$$

and

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = 0 \quad -(14)$$

Define:

$$u = 1/r \quad -(15)$$

then

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta}$$

$$= -\frac{1}{r^2}\frac{\dot{r}^2}{\dot{\theta}^2} \quad -(16)$$

3)

Now use:

$$\dot{\theta} = \frac{L}{mr^2} \quad -(17)$$

So:

$$\frac{du}{d\theta} = -\frac{m}{L}\dot{r} \quad -(18)$$

Therefore:

$$\frac{d^2u}{d\theta^2} = \frac{d}{d\theta}\left(\frac{du}{d\theta}\right) = \frac{dt}{d\theta}\left(-\frac{m}{L}\dot{r}\right)$$

$$= -\frac{m}{L}\frac{\ddot{r}}{\dot{\theta}} \quad -(19)$$

From eqs. (17) and (19):

$$\frac{d^2u}{d\theta^2} = -\frac{m^2}{L^2}r^2\ddot{r} \quad -(20)$$

From eq. (14):

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{\partial U}{\partial r} = F(r) \quad -(21)$$

In this equation:

$$\ddot{r} = -\frac{L^2}{m^2}u^2\frac{d^2u}{d\theta^2} \quad -(22)$$

$$r\dot{\theta}^2 = \frac{L^2}{m^2}u^3 \quad -(23)$$

So

$$\boxed{\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2}\frac{1}{u^2}F(u)} \quad -(24)$$

The ellipse is given from eq. (24) by:

4)

$$F(u) = -mMGu^2 \quad -(25)$$

so

$$\frac{d^2u}{d\theta^2} + u = \frac{Gm^2M}{L^2} = \frac{1}{\alpha} \quad -(26)$$

It is claimed in Einsteinian general relativity that:

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{\alpha} + \delta u^2 \quad -(27)$$

but the evaluation of this equation by computer gives a precessing ellipse only in the case:

$$\frac{\delta}{\alpha^2}\left(1+\frac{\epsilon^2}{2}\right) - \frac{\delta\epsilon^2}{6\alpha^2}\cos 2\theta = 0 \quad -(28)$$

The correct evaluation of Einstein's theory does not give an ellipse at all in general.