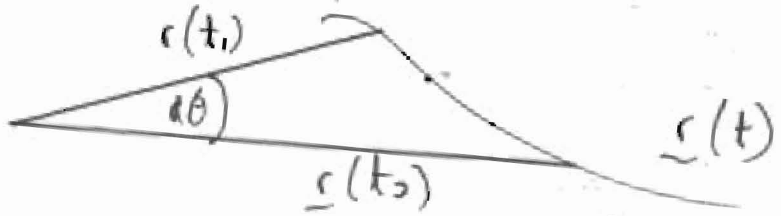
1)

190(2): Modification to Kepler's Second Law.

Consider the path $\underset{\sim}{r}(t)$ as in Fig. (1):




The radius vector and area $A$ are related by:

$$dA = \frac{1}{2} r^2 \, d\theta \qquad -(1)$$

The areal velocity is:

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} \qquad -(2)$$

From eq. (4) of note 190(1):

$$\omega = \frac{d\theta}{dt} = \frac{c^2 L}{E}\left(\frac{m(r)}{r^2}\right) \qquad -(3)$$

where

$$x = \frac{c^2 L}{E} \qquad -(4)$$

is a constant of motion. Therefore:

$$\boxed{\frac{dA}{dt} = \frac{1}{2} x \, m(r)} \qquad -(5)$$

This equation is true for any type of planar orbit observed in astronomy, and is the modification of Kepler's Second Law (1609).

The angular velocity for any orbit is:

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$$\frac{d\theta}{dt} = \omega = \frac{c^2 L}{E} \frac{m(r)}{r^2} \quad - (6)$$

i.e

$$\boxed{\omega = \frac{d\theta}{dt} = x \frac{m(r)}{r^2}} \quad - (7)$$

The quantities $dA/dt$ and $d\theta/dt$ can be measured with accuracy in contemporary astronomy, so the function $m(r)$ can be determined experimentally.

Newtonian Limit

This is described by:

$$m(r) \rightarrow 1 \quad - (8)$$

$$x \rightarrow \frac{L}{m} \quad - (9)$$

where $m$ is the mass of the attracted object such as a planet. So in Newtonian dynamics:

$$\omega = \left(\frac{L}{m}\right) \frac{1}{r^2} \quad - (10)$$

and the force of attraction between $m$ and $M$ is:

$$F = - \frac{mMG}{r^2}, \quad - (11)$$

so in Newtonian dynamics:

$$F = - \left(\frac{m^2 M G}{L}\right) \omega. \quad - (12)$$

In Newtonian dynamics the force is directly

3)

proportional to the angular velocity, and the result is a stable elliptical orbit.

This is true if and only if eq. (8) is true. Otherwise the orbit is not a stable ellipse. For example, if $m(r)$ is approximated by:

$$m(r) \sim 1 - \frac{r_0}{r} \quad -(13)$$

then Kepler's second law is modified from the original:

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{L}{2m} \quad -(14)$$

$$= \text{constant}$$

to:

$$\boxed{\frac{dA}{dt} = \frac{1}{2}\left(\frac{c^2 L}{E}\right)\left(1 - \frac{r_0}{r}\right)} \quad -(15)$$

where

$$r_0 = \frac{2MG}{c^2} \quad -(16)$$

In binary pulsars:

$$\frac{dA}{dt} = \frac{1}{2}\left(\frac{c^2 L}{E}\right)\left(1 - \frac{r_0}{r} - \frac{a}{r^2}\right) -(17)$$

and in general:

$$\boxed{\frac{dA}{dt} = \frac{1}{2}\left(\frac{c^2 L}{E}\right) \exp\left(2 \exp\left(-\frac{r}{R}\right)\right)} -(18)$$

It would be interesting to search the literature in astronomy to find whether Kepler's second law has ever been used as a test of Einsteinian general relativity, eq. (15). A second test would be to measure the angular velocity of an orbiting object and compare with:

$$\omega = \frac{d\theta}{dt} = \left(\frac{c^2 L}{E}\right) \frac{1}{r^2}\left(1 - \frac{r_0}{r}\right) \quad -(19)$$

for Einsteinian general relativity. In general:

$$\omega = \frac{d\theta}{dt} = \left(\frac{c^2 L}{E}\right) \frac{1}{r^2} \exp\left(2\exp\left(-\frac{r}{R}\right)\right) \quad -(20)$$

in which $c^2L / E$ is a constant.

If $m(r)$ is approximated by a polynomial then in general, Kepler's second law is:

$$\boxed{\frac{dA}{dt} = \frac{1}{2}\left(\frac{c^2 L}{E}\right)\left(1 - \frac{r_0}{r} - \frac{a}{r^2} - \frac{b}{r^3} - \ldots\right)} \quad -(21)$$

d

$$ds^2 = c^2 d\tau^2 = m(r)\, c^2 dt^2 - m^{-1}(r)\, dr^2 - r^2 d\theta^2 \quad -(22)$$

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$$m(r) = 1 - \frac{r_0}{r} - \frac{a}{r^2} - \frac{b}{r^3} - \ldots \quad -(23)$$

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