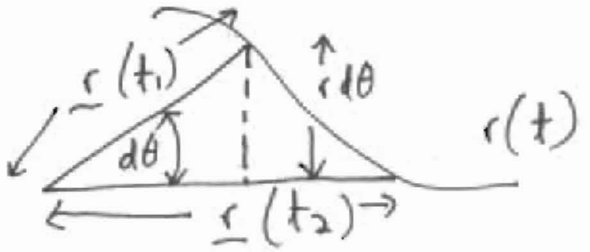
190(15): Accurate Derivation of Kepler's Second Law.

The usual textbook derivation is



i.e.

$$x = \underline{r}(t_1)\sin(d\theta) \quad -(1)$$

$$\sim r\, d\theta$$

so the area of the triangle is:

$$dA_1 = \frac{1}{2} r^2 d\theta. \quad -(2)$$

A flat two dimensional space is used and the area element in (2) is:

$$dA_1 = \int_0^r r\, dr\, d\theta = \frac{1}{2} r^2 d\theta \quad -(3)$$

The true area element of the cylindrical polar system is:

$$dA_2 = r\, dr\, d\theta. \quad -(4)$$

The textbooks assert that Kepler's second law is:

$$\frac{dA_1}{d\theta} = \frac{1}{2} r^2. \quad -(5)$$

For a planar orbit the lagrangian is:

$$\mathcal{L} = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \quad -(6)$$

in cylindrical polar coordinates. The angular momentum is

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m r^2 \frac{d\theta}{dt} = \text{constant}. \quad -(7)$$

This follows from the fact that the angular momentum L is described by the conjugate to the coordinate θ in the Euler Lagrange equation:

2)

$$\dot{L} = \frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 0 \quad -(8)$$

From eqs. (5) and (7):

$$\frac{dA_1}{dt} = \frac{L}{2m} = \text{constant} \quad -(9)$$

This is asserted to be the equal areas in equal times law and to be valid for any closed orbit like an ellipse.

<u>However</u>, direct integration of eq. (3) gives:

$$A = \frac{1}{2} r^2 \int_0^{2\pi} d\theta = \pi r^2 \quad -(10)$$

and <u>this</u> is <u>the area</u> of a <u>circle</u>, <u>not an ellipse</u>. The area of an ellipse is:

$$A = ab\pi \quad -(11)$$

where

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad -(12)$$

i.e.

$$A = \int_{-a}^{a} \int_{-b(a^2-x^2)^{1/2}/a}^{b(a^2-x^2)^{1/2}/a} dy\, dx \quad -(13)$$

$$= \int_{-a}^{a} \frac{2b}{a}\left(a^2 - x^2\right)^{1/2} dx$$

$$= \frac{2b}{a}\left[\frac{1}{2}\left[x\left(a^2-x^2\right)^{1/2} + a^2 \sin^{-1}\left(\frac{x}{|a|}\right)\right]\right]_{-a}^{a}$$

$$= ab\pi$$

3)

This suggests that eq. (5) is not a general equation, even in Newtonian mechanics.

The general method must be based on the infinitesimal line element:

$$ds^2 = d\underline{r} \cdot d\underline{r} = dr^2 + r^2 d\theta^2 \quad -(14)$$

in the plane:

$$dz = 0. \quad -(15)$$

Thus for a given parameter λ:

$$\left(\frac{ds}{d\lambda}\right)^2 = \left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\theta}{d\lambda}\right)^2 \quad -(16)$$

and

$$S = \int_{S_1}^{S_2} \left(\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\theta}{d\lambda}\right)^2\right)^{1/2} d\lambda$$

is the arc length from S_1 to S_2.

The circle may be parameterized by:

$$\lambda = \theta \quad -(17)$$

$$X = r\cos\theta, \quad Y = r\sin\theta \quad -(18)$$

$$X^2 + Y^2 = r^2 = \text{constant} \quad -(18)$$

so

$$S = \int_0^{2\pi} \left(\left(\frac{dr}{d\theta}\right)^2 + r^2\left(\frac{d\theta}{d\theta}\right)^2\right)^{1/2} d\theta,$$

$$S = r\int_0^{2\pi} d\theta = 2\pi r \quad -(19)$$

and the arc length is the circumference of the circle.

4) Note that:

$$\frac{dr}{d\theta} = 0, \quad \frac{d\theta}{d\theta} = 1 \quad -(20)$$

for a circle, whose area is.

$$A = \int_0^r s \, dr = \pi r^2 \quad -(21)$$

However, for an ellipse:

$$X = a\cos\theta, \quad Y = b\sin\theta \quad -(22)$$

and

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1. \quad -(23)$$

So the arc length of the ellipse is:

$$s = \int_0^{2\pi} \left(\left(\frac{dr}{d\theta} \right)^2 + r^2 \right)^{1/2} d\theta \quad -(24)$$

In cylindrical polar coordinates the ellipse is:

$$r = \frac{\alpha}{1 + \epsilon\cos\theta} \quad -(25)$$

where 2α is the latus rectum and ϵ the eccentricity:

$$a = \frac{\alpha}{1-\epsilon^2}, \quad b = \frac{\alpha}{(1-\epsilon^2)^{1/2}}. \quad -(26)$$

$$\frac{dr}{d\theta} = \frac{\epsilon\alpha\sin\theta}{(1+\epsilon\cos\theta)^2} \quad -(27)$$

so the circumference of the ellipse is the arc length:

$$S = \int_0^{2\pi} \left[\left(\frac{\alpha}{1+\epsilon\cos\theta} \right)^2 + \left(\frac{\epsilon\alpha\sin\theta}{(1+\epsilon\cos\theta)^2} \right)^2 \right]^{1/2} d\theta \quad -(28)$$

$$= \int_0^{2\pi} \left[\left(\frac{\alpha}{1+\epsilon\cos\theta} \right)^2 \left[1 + \epsilon^2 \sin^2\theta \left(\frac{\alpha}{1+\epsilon\cos\theta} \right)^2 \right] \right]^{1/2} d\theta$$

$-(28)$

It can be seen that the circumference is calculated from ds^2. Similarly the area element of the ellipse is calculated from:

$$\boxed{dA = (dr^2)^{1/2} (r^2 d\theta^2)^{1/2}} \quad -(29).$$

and this is the correct general definition of area in cylindrical polar coordinates. It originates again from:

$$ds^2 = dr^2 + r^2 d\theta^2 \quad -(30)$$

For a circle:

$$A = \int_0^{2\pi} \int_0^{r_0} r\,dr\,d\theta = \pi r_0^2 \quad -(31)$$

and

$$\frac{dr}{d\theta} = 0. \quad -(32)$$

However, for an ellipse:

$$dr = \frac{\epsilon\alpha\sin\theta}{(1+\epsilon\cos\theta)^2} d\theta \quad -(33)$$

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(34)$$

6) and from eqs. (29), (33) and (34):

$$dA = \frac{\epsilon \alpha \sin\theta}{(1+\epsilon\cos\theta)^2} d\theta \cdot r\, d\theta \quad -(35)$$

$$\boxed{dA = \frac{\epsilon \alpha^2 \sin\theta}{(1+\epsilon\cos\theta)^3} (d\theta)^2} \quad -(36)$$

The area is:

$$A = \int_0^{2\pi} \left(\int r\, dr \right) d\theta \quad -(37)$$

and

$$dA_1 = \left(\int_{\theta_2} r\, dr \right) d\theta \quad -(38)$$

where

$$\int r\, dr = \int_{\theta_1} \frac{\epsilon \alpha^2 \sin\theta}{(1+\epsilon\cos\theta)^3} d\theta \quad -(39)$$

Conclusion

Kepler's law for an ellipse is:

$$\boxed{\frac{dA_1}{d\theta} = \int_{\theta_1}^{\theta_2} \frac{\epsilon \alpha^2 \sin\theta}{(1+\epsilon\cos\theta)^3} d\theta} \quad -(40)$$

which does not appear to be the same as eq (5).

7)

The other major problem is that Einsteinian GR has not accounted for the fact that in curved spacetime:

$$ds^2 = \frac{dr^2}{m(r)} + r^2 d\theta^2 \quad -(41)$$

So:

$$\boxed{A = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} \frac{r \, dr}{(m(r))^{1/2}} d\theta} \quad -(42)$$

and

$$\frac{dA_1}{d\theta} = \int_{r_1}^{r_2} \frac{r}{(m(r))^{1/2}} dr \quad -(43)$$

Even for the circle, the result is not the same as eq. (5), i.e.

$$\frac{dA_1}{d\theta} = \int_0^{r_0} \frac{r}{m^{1/2}} dr \quad -(44)$$

For example, if:

$$m^{-1/2} = \left(1 - \frac{r_0}{r}\right)^{-1/2}$$

$$\frac{dA_1}{d\theta} = \int_0^{R_0} r \left(1 - \frac{r_0}{r}\right)^{-1/2} dr$$

$$\sim \int_0^{R_0} r \left(1 + \frac{1}{2}\frac{r_0}{r}\right) dr = \frac{1}{2}\left(R_0^2 + r_0 R_0\right)$$

$$\frac{dA_1}{d\theta} = \frac{1}{2}\left(R_0^2 + \frac{2mGR_0}{c^2}\right) \quad -(45)$$