1) 112(2): Duality Invariance of the Bianchi Identity.

This is the key geometrical property of Cartan geometry that gives the field equations of physics. It is therefore helpful to give proofs and examples of the theorem in the clearest way possible. If for example the following is considered:

$$[D_2, D_3]V^\kappa = R^\kappa{}_{\sigma 23}V^\sigma - T^\lambda{}_{23}D_\lambda V^\kappa, \quad -(1)$$

its Hodge dual is:

$$[D^0, D^1]_{HD}V^\kappa = \tilde{R}^\kappa{}_\sigma{}^{01}V^\sigma - \tilde{T}^{\lambda 01}D_\lambda V^\kappa. \quad -(2)$$

Lowering indices:

$$[D_0, D_1]_{HD}V^\kappa = \tilde{R}^\kappa{}_{\sigma 01}V^\sigma - \tilde{T}^\lambda{}_{01}D_\lambda V^\kappa \quad -(3)$$

Indices are lowered by use of the metric as usual:

$$[D_0, D_1]_{HD} = g_{00}\, g_{11}\, [D^0, D^1]_{HD} \quad -(4)$$

$$\tilde{R}^\kappa{}_{\sigma 01} = g_{00}\, g_{11}\, \tilde{R}^\kappa{}_\sigma{}^{01} \quad -(5)$$

$$\tilde{T}^\lambda{}_{01} = g_{00}\, g_{11}\, \tilde{T}^{\lambda 01} \quad -(6)$$

so eq. (3) follows from eq. (2). So in general, if

$$[D_\mu, D_\nu]V^\kappa = R^\kappa{}_{\sigma\mu\nu}V^\sigma - T^\lambda{}_{\mu\nu}D_\lambda V^\kappa \quad -(7)$$

then:

$$[D_\mu, D_\nu]_{HD}V^\kappa = \tilde{R}^\kappa{}_{\sigma\mu\nu}V^\sigma - \tilde{T}^\lambda{}_{\mu\nu}D_\lambda V^\kappa. \quad -(8)$$

Eq. (7) is:

$$[D_\mu, D_\nu]V^\rho = D_\mu(D_\nu V^\rho) - D_\nu(D_\mu V^\rho) \quad -(9)$$

2)

$$= R^{\rho}{}_{\sigma\mu\nu} V^{\sigma} - T^{\lambda}_{\mu\nu} D_{\lambda} V^{\rho}.$$

It follows from eq. (9) that if:

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \quad -(10)$$

then:

$$T^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}. \quad -(11)$$

Given the tetrad postulate:

$$D_{\mu} q^{a}_{\nu} = \partial_{\mu} q^{a}_{\nu} + \omega^{a}_{\mu b} q^{b}_{\nu} - \Gamma^{\lambda}_{\mu\nu} q^{a}_{\lambda}$$
$$= 0 \quad -(12)$$

then eqs (10) and (11) imply:

$$\boxed{D \wedge T^{a} := R^{a}{}_{b} \wedge q^{b}.} \quad -(13)$$

In eq. (9):

$$D_{\nu} V^{\rho} = \partial_{\nu} V^{\rho} + \Gamma^{\rho}_{\nu\lambda} V^{\lambda} \quad -(14)$$
$$D_{\mu} V^{\rho} = \partial_{\mu} V^{\rho} + \Gamma^{\rho}_{\mu\lambda} V^{\lambda}. \quad -(15)$$

Similarly, it is possible to write eq. (8) as:

$$[D_{\mu}, D_{\nu}]_{HD} V^{\rho} = D_{\mu}(D_{\nu} V^{\rho}) - D_{\nu}(D_{\mu} V^{\rho}) \quad -(16)$$

where:

$$D_{\nu} V^{\rho} = \partial_{\nu} V^{\rho} + \Lambda^{\rho}_{\nu\lambda} V^{\lambda} \quad -(17)$$
$$D_{\mu} V^{\rho} = \partial_{\mu} V^{\rho} + \Lambda^{\rho}_{\mu\lambda} V^{\lambda} \quad -(18)$$

3)

So:

$$\tilde{R}^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Lambda^{\rho}_{\nu\sigma} - \partial_\nu \Lambda^{\rho}_{\mu\sigma} + \Lambda^{\rho}_{\mu\lambda}\Lambda^{\lambda}_{\nu\sigma} - \Lambda^{\rho}_{\nu\lambda}\Lambda^{\lambda}_{\mu\sigma} \qquad -(19)$$

and $$\tilde{T}^{\lambda}{}_{\mu\nu} = \Lambda^{\lambda}_{\mu\nu} - \Lambda^{\lambda}_{\nu\mu} \,. \qquad -(20)$$

Given the tetrad postulate:

$$D_\mu q^a_\nu = \partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\nu - \Lambda^{\lambda}_{\mu\nu} q^a_\lambda \qquad -(21)$$

it follows that:

$$\boxed{D \wedge \tilde{T}^a := \tilde{R}^a{}_b \wedge q^b.} \qquad -(22)$$

So eqns. (13) and (22) follow directly from the fact that $[D_\mu, D_\nu]$ and $[D_\mu, D_\nu]_{HD}$ are antisymmetric in μ and ν.