Relativistic Cosmology Revisited

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In a previous paper the writer treated of particular classes of cosmological solutions for certain Einstein spaces and claimed that no such solutions exist in relation thereto. In that paper the assumption that the proper radius is zero when the line-element is singular was generally applied. This general assumption is unjustified and must be dropped. Consequently, solutions do exist in relation to the aforementioned types, and are explored herein. The concept of the Big Bang cosmology is found to be inconsistent with General Relativity.

1. Introduction

In a previous paper [1] the writer considered what he thought was a general problem statement in relation to certain Einstein spaces, and concluded that no such solutions exist for those types. However, the problem statement treated in the aforementioned paper adopted an unjustified assumption - that the proper radius is zero when the line-element is singular. Although this occurs in the case of the gravitational field for $R_{\mu\nu} = 0$, it is not a general principle and so it cannot be generally applied, even though it can be used to amplify various errors in the usual analysis of the well known cosmological models, as done in [1]. By dropping the assumption it is found that cosmological solutions do exist, but none are consistent with the alleged Big Bang cosmology.

2. The so-called Schwarzschild-de Sitter model

Consider the line-element

$$ds^2 = \left(1 - \frac{\alpha}{R_c} - \frac{\lambda}{3}R_c^2\right)dt^2 -$$

$$-\left(1 - \frac{\alpha}{R_c} - \frac{\lambda}{3}R_c^2\right)^{-1} dR_c^2 - R_c^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \quad (1)$$

where $R_c = R_c(r)$ is the radius of curvature, r a parameter, and α a function of mass. This has no solution for some function $R_c(r)$ on $R_c(r) \to \infty$ [1].

If $\alpha = 0$, (1) reduces to

$$ds^2 = \left(1 - \frac{\lambda}{3}R_c^2\right)dt^2 -$$

$$-\left(1 - \frac{\lambda}{3}R_c^2\right)^{-1} dR_c^2 - R_c^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \quad (2)$$

This has no solution for some function $R_c(r)$ on $\sqrt{\frac{3}{\lambda}} < R_c(r) < \infty$ [1].

For $1 - \frac{\lambda}{3}R_c^2 > 0$ and $R_c \geq 0$, it is required that

$$0 \leq R_c < \sqrt{\frac{3}{\lambda}}. \quad (3)$$

The proper radius on (2) is

$$R_p = \int \frac{dR_c}{\sqrt{1 - \frac{\lambda}{3}R_c^2}} = \sqrt{\frac{3}{\lambda}}\arcsin\sqrt{\frac{\lambda}{3}R_c^2} + K,$$

where K is a constant. $R_p = 0$ is satisfied if $R_c = 0 = K$, in accord with (3). Then

$$R_p = \sqrt{\frac{3}{\lambda}}\arcsin\sqrt{\frac{\lambda}{3}R_c^2}.$$

Now

$$\sqrt{\frac{3}{\lambda}}\arcsin 1 = \sqrt{\frac{3}{\lambda}}\frac{(1+4n)\,\pi}{2}$$

$$= \lim_{R_c \to \sqrt{\frac{3}{\lambda}}^-} \sqrt{\frac{3}{\lambda}}\arcsin\sqrt{\frac{\lambda}{3}}R_c = \lim_{R_c \to \sqrt{\frac{3}{\lambda}}^-} R_p,$$

$$n = 0, 1, 2, ... \quad (4)$$

in accord with (3). Thus, R_p can be arbitrarily large. Moreover, R_p can be arbitrarily large for any R_c satisfying (3) since

$$R_p = \sqrt{\frac{3}{\lambda}}\arcsin\sqrt{\frac{\lambda}{3}}R_c = \sqrt{\frac{3}{\lambda}}\left(\psi + 2n\pi\right),$$

$$n = 0, 1, 2, ...$$

where ψ is in radians, $0 \leq \psi < \frac{\pi}{2}$.

In the case of (1), the mutual constraints on the radius of curvature are

$$\frac{\lambda}{3}R_c^3 - R_c + \alpha < 0$$

$$0 < R_c(r). \tag{5}$$

The proper radius on (1) is

$$R_p(r) = \int \frac{dR_c}{\sqrt{1 - \frac{\alpha}{R_c} - \frac{\lambda}{3}R_c^2}} + K, \tag{6}$$

where K is a constant, subject to $R_p \geq 0$. The difficulty here is the cubic in (5) and (6). The approximate positive roots to the cubic are α and $\sqrt{\frac{3}{\lambda}}$. These must correspond to limiting values in the integral (6). Both $R_c(r)$ and $R_p(r)$ also contain α and λ.

In addition, it was argued in [1] that the admissible form for $R_c(r)$ in (1) must reduce, when $\lambda = 0$, to the Schwarzschild form

$$R_c(r) = \left(|r - r_0|^n + \alpha^n\right)^{\frac{1}{n}}$$
$$n \in \Re^+, \quad r \in \Re, \quad r \neq r_0, \tag{7}$$

where r_0 and n are entirely arbitrary constants. Note that when $\alpha = 0$ and $\lambda = 0$, (1) reduces to Minkowski space and (7) reduces to the radius of curvature in Minkowski space, as necessary.

Determination of the required general parametric expression for $R_c(r)$ in relation to (1), having all the required properties, is not a simple problem. Numerical methods suggest however [1], that there may in fact be no solution for $R_c(r)$ in relation to (1), subject to the stated constraints. At this time the question remains open.

3. Einstein's cylindrical model

Consider the line-element

$$ds^2 = dt^2 - \left[1 - (\lambda - 8\pi P_0)\, R_c^2\right]^{-1} dR_c^2 - $$
$$-R_c^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{8}$$

This of course has no Lorentz signature solution in $R_c(r)$ for $\frac{1}{\sqrt{\lambda - 8\pi P_0}} < R_c(r) < \infty$ [1].

For $1 - (\lambda - 8\pi P_0)\, R_c^2 > 0$ and $R_c = R_c(r) \geq 0$,

$$0 \leq R_c < \frac{1}{\sqrt{\lambda - 8\pi P_0}}. \tag{9}$$

The proper radius is

$$R_p = \int \frac{dR_c}{\sqrt{1 - (\lambda - 8\pi P_0)\, R_c^2}}$$
$$= \frac{1}{\sqrt{\lambda - 8\pi P_0}} \arcsin\sqrt{(\lambda - 8\pi P_0)\, R_c^2} + K,$$

where K is a constant. $R_p = 0$ is satisfied for $R_c = 0 = K$, so that

$$R_p = \frac{1}{\sqrt{\lambda - 8\pi P_0}} \arcsin\sqrt{(\lambda - 8\pi P_0)\, R_c^2},$$

in accord with (9).

Now

$$\frac{1}{\sqrt{\lambda - 8\pi P_0}} \arcsin 1 = \frac{(1 + 4n)\,\pi}{2\sqrt{\lambda - 8\pi P_0}}$$
$$= \lim_{R_c \to \frac{1}{\sqrt{\lambda - 8\pi P_0}}^-} \frac{1}{\sqrt{\lambda - 8\pi P_0}} \arcsin\sqrt{(\lambda - 8\pi P_0)\, R_c^2}$$
$$n = 0, 1, 2, ...$$

in accord with (9). Thus R_p can be arbitrarily large. Moreover, R_p can be arbitrarily large for any R_c satisfying (9), since

$$R_p = \frac{1}{\sqrt{\lambda - 8\pi P_0}} \arcsin\sqrt{(\lambda - 8\pi P_0)\, R_c^2} = \frac{(\psi + 2n\pi)}{\sqrt{\lambda - 8\pi P_0}},$$
$$n = 0, 1, 2, ...$$

where ψ is in radians, $0 \leq \psi < \frac{\pi}{2}$.

4. de Sitter's spherical model

Consider the line-element

$$ds^2 = \left(1 - \frac{\lambda + 8\pi\rho_{00}}{3} R_c^2\right) dt^2 -$$
$$-\left(1 - \frac{\lambda + 8\pi\rho_{00}}{3} R_c^2\right)^{-1} dR_c^2 -$$
$$-R_c^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{10}$$

This has no Lorentz signature solution in some $R_c(r)$ on $\sqrt{\frac{3}{\lambda + 8\pi\rho_{00}}} < R_c(r) < \infty$ [1].

For $1 - \frac{\lambda + 8\pi\rho_{00}}{3} R_c^2 > 0$ and $R_c = R_c(r) \geq 0$,

$$0 \leq R_c < \sqrt{\frac{3}{\lambda + 8\pi\rho_{00}}}. \tag{11}$$

The proper radius is

$$R_p = \int \frac{dR_c}{\sqrt{\left(1 - \frac{\lambda + 8\pi\rho_{00}}{3}\right) R_c^2}}$$
$$= \sqrt{\frac{3}{\lambda + 8\pi\rho_{00}}} \arcsin\sqrt{\left(\frac{\lambda + 8\pi\rho_{00}}{3}\right) R_c^2} + K,$$

where K is a constant. $R_p = 0$ is satisfied for $R_c = 0 = K$, so

$$R_p = \sqrt{\frac{3}{\lambda + 8\pi\rho_{00}}} \arcsin\sqrt{\left(\frac{\lambda + 8\pi\rho_{00}}{3}\right) R_c^2},$$

in accord with (11).

Now

$$\sqrt{\frac{3}{\lambda+8\pi\rho_{00}}}\arcsin 1=\sqrt{\frac{3}{\lambda+8\pi\rho_{00}}}\frac{(1+4n)\,\pi}{2}$$

$$=\lim_{R_c\to\sqrt{\frac{3}{\lambda+8\pi\rho_{00}}}^-}\sqrt{\frac{3}{\lambda+8\pi\rho_{00}}}\arcsin\sqrt{\left(\frac{\lambda+8\pi\rho_{00}}{3}\right)R_c^2},$$

$$n=0,1,2,...$$

in accord with (11). Thus R_p can be arbitrarily large. Moreover, R_p can be arbitrarily large for any R_c satisfying (11), since

$$R_p=\sqrt{\frac{3}{\lambda+8\pi\rho_{00}}}\arcsin\sqrt{\left(\frac{\lambda+8\pi\rho_{00}}{3}\right)R_c^2}$$

$$=\sqrt{\frac{3}{\lambda+8\pi\rho_{00}}}\left(\psi+2n\pi\right),$$

$$n=0,1,2,...$$

where ψ is in radians, $0\le\psi<\frac{\pi}{2}$.

5. Cosmological models of expansion

Transform (10) by

$$\bar{R}_c=\frac{R_c}{\sqrt{1-\frac{R_c^2}{W^2}}}e^{-\frac{t}{W}},\quad \bar{t}=t+\frac{1}{2}W\ln\left(1-\frac{R_c^2}{W^2}\right),$$

$$W^2=\frac{3}{\lambda+8\pi\rho_{00}},$$

to get

$$ds^2=d\bar{t}^2-e^{\frac{2\bar{t}}{W}}(d\bar{R}_c^2+\bar{R}_c^2d\theta^2+\bar{R}_c^2\sin^2\theta d\varphi^2), \qquad (12)$$

where according to (11), $0\le\bar{R}_c<\infty$. Clearly the proper radius on (12) is

$$\bar{R}_p=\lim_{\bar{R}_c\to\infty}e^{\frac{\bar{t}}{W}}\int_0^{\bar{R}_c}d\bar{R}_c=\infty.$$

Therefore (12) describes an infinite Universe for all $\bar{t}$.

Consider the line-element

$$ds^2=dt^2-\frac{e^{g(t)}}{\left(1+\frac{k}{4}G^2\right)^2}\left[dG^2+G^2\left(d\theta^2+\sin^2\theta d\varphi^2\right)\right], \qquad (13)$$

where $G=G(r)$, r a parameter. If $k=0$ a form of (12) is obtained. If $k>0$,

$$R_p=e^{\frac{1}{2}g(t)}\int\frac{dG}{1+\frac{k}{4}G^2}$$

$$=e^{\frac{1}{2}g(t)}\left[\frac{2}{\sqrt{k}}\arctan\frac{\sqrt{k}}{2}G+K\right],$$

where K is a constant. $R_p=0$ is satisfied by $G=0=K$, so

$$R_p=e^{\frac{1}{2}g(t)}\int\frac{dG}{1+\frac{k}{4}G^2}=e^{\frac{1}{2}g(t)}\frac{2}{\sqrt{k}}\arctan\frac{\sqrt{k}}{2}G.$$

Now for (13), the radius of curvature is

$$R_c=\frac{G}{1+\frac{k}{4}G^2}, \qquad (14)$$

which is maximum when $G=\frac{2}{\sqrt{k}}$, i.e.

$$R_{c_{max}}=R_c(\frac{2}{\sqrt{k}})=\frac{1}{\sqrt{k}}.$$

Also, $\lim_{G\to\infty}R_c=0$. Therefore, on (13),

$$0\le R_c\le\frac{1}{\sqrt{k}}, \qquad (15)$$

or equivalently

$$0\le G\le\frac{2}{\sqrt{k}}. \qquad (16)$$

Now

$$R_p\left(G=\frac{2}{\sqrt{k}}\right)=e^{\frac{1}{2}g(t)}\arctan 1=e^{\frac{1}{2}g(t)}\arctan 1$$

$$=e^{\frac{1}{2}g(t)}\frac{(1+4n)\,\pi}{4},$$

$$n=0,1,2,...$$

which is arbitrarily large. Moreover, R_p is arbitrarily large for any R_c satisfying (15) (or equivalently for any G satisfying (16)), since

$$R_p=e^{\frac{1}{2}g(t)}\frac{2}{\sqrt{k}}\left(\psi+n\pi\right),\quad n=0,1,2,...$$

where ψ is in radians, $0\le\psi\le\frac{\pi}{4}$.

If $k<0$, set $k=-s,\ s>0$. Then

$$R_p=e^{\frac{1}{2}g(t)}\int\frac{dG}{1-\frac{s}{4}G^2}$$

$$=e^{\frac{1}{2}g(t)}\left[\frac{1}{\sqrt{s}}\ln\left|\frac{G+\frac{2}{\sqrt{s}}}{G-\frac{2}{\sqrt{s}}}\right|+K\right],$$

where K is a constant. $R_p=0$ is satisfied for $G=0=K$. Then

$$R_p=e^{\frac{1}{2}g(t)}\frac{1}{\sqrt{s}}\ln\left|\frac{G+\frac{2}{\sqrt{s}}}{G-\frac{2}{\sqrt{s}}}\right|.$$

To maintain signature in (13),

$$-\frac{2}{\sqrt{s}}<G<\frac{2}{\sqrt{s}}.$$

However, since a negative radius of curvature is meaningless, and since on (13) the radius of curvature in this case is

$$R_c(G) = \frac{G}{1 - \frac{s}{4}G^2}, \tag{17}$$

it is required that

$$0 \leq G < \frac{2}{\sqrt{s}}. \tag{18}$$

Now

$$\lim_{G \to \frac{2}{\sqrt{s}}^-} e^{\frac{1}{2}g(t)} \frac{1}{\sqrt{s}} \ln \left| \frac{G + \frac{2}{\sqrt{s}}}{G - \frac{2}{\sqrt{s}}} \right| = \infty,$$

in accord with (18). The proper radius of the space and the radius of curvature of the space are therefore infinite for all time t.

The usual transformation of (13) to obtain the Robertson-Walker line-element involves expressing (13) in terms of the radius of curvature of (13) instead of the quantity G, thus

$$\bar{G} = \frac{G}{1 + \frac{k}{4}G^2},$$

carrying (13) into

$$ds^2 = dt^2 - e^{g(t)} \left[\frac{d\bar{G}^2}{1 - k\bar{G}^2} + \bar{G}^2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right) \right]. \tag{19}$$

If $k = 0$ a form of (12) is obtained.

Comparing $\bar{G}$ with (14) it is plain that $\bar{G} = R_c(G)$, where $0 \leq R_c \leq \frac{1}{\sqrt{k}}$ by (15), $k > 0$, and therefore $0 \leq \bar{G} \leq \frac{1}{\sqrt{k}}$. Now

$$R_p = e^{\frac{1}{2}g(t)} \int \frac{dR_c}{\sqrt{1 - kR_c^2}}$$

$$= e^{\frac{1}{2}g(t)} \left(\frac{1}{\sqrt{k}} \arcsin \sqrt{k} R_c + K \right),$$

where K is a constant. $R_p = 0$ is satisfied for $R_c = 0 = K$, so

$$R_p = e^{\frac{1}{2}g(t)} \frac{1}{\sqrt{k}} \arcsin \sqrt{k} R_c,$$

in accord with (15).

Then

$$R_p(R_c = \frac{1}{\sqrt{k}}) = e^{\frac{1}{2}g(t)} \frac{1}{\sqrt{k}} \left(\frac{\pi}{2} + 2n\pi \right), \tag{20}$$

$$n = 0, 1, 2, ...$$

in accord with (15), and so R_p is arbitrarily large for all time t. When making the transformation to the Robertson-Walker form the limits on the transformed coordinate cannot be ignored. Moreover, R_P is arbitrarily large for all time for any R_c satisfying (15), since

$$R_p = e^{\frac{1}{2}g(t)} \frac{1}{\sqrt{k}} (\psi + 2n\pi),$$

$$n = 0, 1, 2, ...$$

where ψ is in radians, $0 \leq \psi \leq \frac{\pi}{6}$.

If $k < 0$ set $k = -s$ where $s > 0$, then (19) becomes

$$ds^2 = dt^2 - e^{g(t)} \left[\frac{dR_c^2}{1 + sR_c^2} + R_c^2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right) \right]. \tag{21}$$

The proper radius is

$$R_p = e^{\frac{1}{2}g(t)} \int \frac{dR_c}{\sqrt{1 + sR_c^2}}$$

$$= e^{\frac{1}{2}g(t)} \left[\frac{1}{\sqrt{s}} \ln \left(R_c + \sqrt{R_c^2 + \frac{1}{s}} \right) + K \right],$$

where K is a constant. $R_p = 0$ is satisfied for $R_c = 0$ and $K = -\frac{1}{\sqrt{s}} \ln \frac{1}{\sqrt{s}}$, in accord with (17) and (18). So

$$R_p = e^{\frac{1}{2}g(t)} \frac{1}{\sqrt{s}} \ln \left(\frac{R_c + \sqrt{R_c^2 + \frac{1}{s}}}{\frac{1}{\sqrt{s}}} \right).$$

Now $R_p \to \infty$ as $R_c \to \infty$, in accord with (17) and (18). Thus, (21) describes an infinite Universe for any time t.

6. Conclusions

By the foregoing types of spacetimes, General Relativity permits cosmological solutions, contrary to the claims made in [1]. However, the Big Bang theory is not consistent with General Relativity, since the spacetimes permitted are all spatially infinite (arbitrarily large) for any time t.

References

[1] Crothers S. J. On the general solution to Einstein's vacuum field for the point-mass when $\lambda \neq 0$ and its consequences for relativistic cosmology. Progress in Physics, v. 3, 7–14, 2005, (www.ptep-online.com).