ANALYTICAL SOLUTION OF THE N PARTICLE GRAVITATIONAL PROBLEM.

by

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ABSTRACT

The first analytical solution is given of the N particle gravitational problem in terms of orbits of pairs of particles together with a novel constraint equation. It is shown that in general, the N particle lagrangian can be factorized into a sum of two particle lagrangians for which the solution is known. Each two particle orbit is inter-related by the constraint equation. In the Newtonian limit each two particle orbit is an ellipse, and more generally it is a precessing conical section of great inherent mathematical richness.

Keywords: Classical limit of ECE theory, solution of the N particle gravitational problem.

1. INTRODUCTION

In recent papers in this series on the applications of ECE theory {1- 10} it has been shown that the precessing conical sections have great inherent mathematical richness on the two particle level in gravitational theory {11}. In this paper the analysis is extended to the well known N particle problem in gravitation, in which one particle interacts with N - 1 others. In Section 2 the first analytical solution of this problem is given by factorizing the lagrangian into N! / ((N - 2)! 2!) equations of two particle orbits. A novel constraint equation is deduced for planar orbits from the fundamental unit vector properties of the planar cylindrical coordinate system. The constraint equation inter-relates the orbits of each pairs of particles, so the orbital motion of one particle depends on the other N - 1 particles. In the solar system such orbits appear to be stable, but even on the Newtonian level the orbits of the N particle problem are in general rich in mathematical structure. The additional consideration of precession as in the immediately preceding papers of this series results in a completely new subject of cosmology. In Section 3 some of the features of the new solution are graphed for illustration. This appears to be the first analytical solution of the N particle gravitational problem obtained in nearly four hundred years.

2. ANALYTICAL SOLUTION

Consider the gravitational interaction of three particles of masses m_1, m_2 and m_3. This is referred to as "the three particle problem". Assume that the particles interact with the Hooke Newton potential {11}. The lagrangian is therefore:

$$\mathcal{L} = \frac{1}{2}\left(m_1|\dot{\underline{r}}_1|^2 + m_2|\dot{\underline{r}}_2|^2 + m_3|\dot{\underline{r}}_3|^2\right) - \frac{m_1 m_2 G}{|\underline{r}_1 - \underline{r}_2|} - \frac{m_1 m_3 G}{|\underline{r}_1 - \underline{r}_3|} - \frac{m_2 m_3 G}{|\underline{r}_2 - \underline{r}_3|} \quad -(1)$$

The radial coordinate of each particle is $\underline{r}_i$, i = 1, 2, 3 and G is Newton's constant. Now note that:

$$\mathcal{L} = \frac{1}{2}\left(\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3\right) \quad -(2)$$

where:

$$\mathcal{L}_1 = \frac{1}{2}\left(m_1|\dot{\underline{r}}_1|^2 + m_2|\dot{\underline{r}}_2|^2\right) - \frac{2m_1m_2G}{|\underline{r}_1 - \underline{r}_2|} \quad -(3)$$

$$\mathcal{L}_2 = \frac{1}{2}\left(m_1|\dot{\underline{r}}_1|^2 + m_3|\dot{\underline{r}}_3|^2\right) - \frac{2m_1m_3G}{|\underline{r}_1 - \underline{r}_3|} \quad -(4)$$

$$\mathcal{L}_3 = \frac{1}{2}\left(m_1|\dot{\underline{r}}_2|^2 + m_3|\dot{\underline{r}}_3|^2\right) - \frac{2m_2m_3G}{|\underline{r}_2 - \underline{r}_3|} \quad -(5)$$

The three particle lagrangian has been factorized into the sum of three two particle lagrangians. Similarly, it can be shown that the four particle lagrangian factorizes into a sum of six two particle lagrangians. In general the N particle lagrangian factorizes into a sum of N! / ((N - 2)! 2!) two particle lagrangians.

The lagrangians (3) to (5) can be written in the format (see notes accompanying UFT219 on www.aias.us):

$$\mathcal{L} = \frac{1}{2}\mu_i|\dot{\underline{R}}_i|^2 - U(R_i). \quad -(6)$$

$$i = 1, 2, 3$$

Here:

$$\mu_1 = \frac{m_1m_2}{m_1+m_2}, \quad \mu_2 = \frac{m_1m_3}{m_1+m_3}, \quad \mu_3 = \frac{m_2m_3}{m_2+m_3}, \quad -(7)$$

and the potential is:

$$U_1 = -\frac{2m_1m_2G}{R_1}, \quad U_2 = -\frac{2m_1m_3G}{R_2}, \quad U_3 = -\frac{2m_2m_3G}{R_3} \qquad -(8)$$

In cylindrical polar coordinates in a plane:

$$\underline{r} = r\,\underline{e}_r \qquad -(9)$$

$$\dot{\underline{r}} = \frac{d\underline{r}}{dt} = \frac{d}{dt}\left(r\,\underline{e}_r\right) \qquad -(10)$$

and the unit vectors of the system are defined by:

$$\underline{e}_r = \underline{i}\cos\theta + \underline{j}\sin\theta \qquad -(11)$$

$$\underline{e}_\theta = -\underline{i}\sin\theta + \underline{j}\cos\theta. \qquad -(12)$$

In Eq. (10):

$$\dot{\underline{r}} = \dot{r}\,\underline{e}_r + r\,\dot{\underline{e}}_r \qquad -(13)$$

and for each particle:

$$\underline{r}_i = r_i\,\underline{e}_r \qquad -(14)$$

$$\dot{\underline{r}}_i = \dot{r}_i\,\underline{e}_r + r_i\,\dot{\underline{e}}_r \qquad -(15)$$

Note carefully that θ does not depend on i, because it is defined by:

$$\dot{\underline{e}}_r = \frac{d\underline{e}_r}{dt} \qquad -(16)$$

where $\underline{e}_r$ is a unit vector. In consequence:

$$\dot{\underline{e}}_r = \dot{\theta}\,\underline{e}_\theta \qquad -(17)$$

and $\dot{\theta}$ does not have an index subscript i. For each particle the Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}_i}{\partial R_i} = \frac{d}{dt}\frac{\partial \mathcal{L}_i}{\partial \dot{R}_i} \quad -(18)$$

$$\frac{\partial \mathcal{L}_i}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}_i}{\partial \dot{\theta}} . \quad -(19)$$

These can be combined into {1 - 11}:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{R_i}\right) + \frac{1}{R_i} = -\frac{\mu_i R_i^2}{L_i^2} F_i(R_i) \quad -(20)$$

in which the conserved angular momentum is:

$$L_i = \mu_i R_i^2 \frac{d\theta}{dt} \quad -(21)$$

and in which the force is:

$$F_i = -\frac{\partial U_i}{\partial R_i} . \quad -(22)$$

The solutions of Eqs. (20) are {1 - 11}:

$$R_i = \frac{\alpha_i}{1 + \epsilon_i \cos\theta} \quad -(23)$$

where:

$$\alpha_i = \frac{L_i^2}{\mu_i k_i}, \quad -(24)$$

$$\epsilon_i = \left(1 + \frac{2E_i L_i^2}{\mu_i k_i^2}\right)^{1/2}, \quad -(25)$$

$$k_1 = 2m_1 m_2 G, \; k_2 = 2m_1 m_3 G, \; k_3 = 2m_2 m_3 G. \quad -(26)$$

Therefore there are three orbits:

$$R_1 = \frac{d_1}{1+\epsilon_1 \cos\theta}, \quad -(27)$$

$$R_2 = \frac{d_2}{1+\epsilon_2 \cos\theta}, \quad -(28)$$

$$R_3 = \frac{d_3}{1+\epsilon_3 \cos\theta}, \quad -(29)$$

for each pair of particles. In obtaining these solutions the centres of mass of each pair of particles are defined by:

$$m_1 \underline{r}_1 + m_2 \underline{r}_2 = \underline{0}, \quad \underline{R}_1 = |\underline{r}_1 - \underline{r}_2|, \quad -(30)$$

$$m_1 \underline{r}_1 + m_3 \underline{r}_3 = \underline{0}, \quad \underline{R}_2 = |\underline{r}_1 - \underline{r}_3|, \quad -(31)$$

$$m_2 \underline{r}_2 + m_3 \underline{r}_3 = \underline{0}, \quad \underline{R}_3 = |\underline{r}_2 - \underline{r}_3|. \quad -(32)$$

From Eqs. (23) the following constraint equation is obtained:

$$\cos\theta = \frac{1}{\epsilon_i}\left(\frac{d_i}{R_i} - 1\right), \quad -(33)$$
$$i = 1, 2, 3,$$

giving three more equations:

$$\frac{1}{\epsilon_1}\left(\frac{d_1}{R_1} - 1\right) = \frac{1}{\epsilon_2}\left(\frac{d_2}{R_2} - 1\right) \quad -(34)$$

$$\frac{1}{\epsilon_1}\left(\frac{d_1}{R_1} - 1\right) = \frac{1}{\epsilon_3}\left(\frac{d_3}{R_3} - 1\right) \quad -(35)$$

$$\frac{1}{\epsilon_2}\left(\frac{d_2}{R_2} - 1\right) = \frac{1}{\epsilon_3}\left(\frac{d_3}{R_3} - 1\right). \quad -(36)$$

To find three more equations use:

$$d_i = \frac{L_i^2}{\mu_i k_i} = \frac{\mu_i r_i^2}{k_i}\frac{d\theta}{dt} \quad -(37)$$

giving:

$$\frac{d_1}{d_2} = \left(\frac{m_1+m_3}{m_1+m_2}\right)\left(\frac{R_1}{R_2}\right)^2 \quad -(38)$$

$$\frac{d_1}{d_3} = \left(\frac{m_2+m_3}{m_1+m_2}\right)\left(\frac{R_1}{R_3}\right)^2, \quad \frac{d_2}{d_3} = \left(\frac{m_2+m_3}{m_1+m_3}\right)\left(\frac{R_2}{R_3}\right)^2 \quad -(39)$$

There are at least nine available equations in the nine unknowns: $R_1, R_2, R_3, d_1, d_2, d_3, \epsilon_1, \epsilon_2, \epsilon_3$ so the problem is soluble analytically, Q.E.D.

For example the solution for R_3 is :

$$R_3 = d_3\left(1 - \frac{\epsilon_3}{\epsilon_2}\left(\frac{d_2 - R_2}{R_2}\right)\right) \quad -(40)$$

where:

$$R_2 = d_2\left(1 - \frac{\epsilon_2}{\epsilon_1}\left(\frac{d_1 - R_1}{R_1}\right)\right) \quad -(41)$$

and

$$R_1 = \frac{d_1}{1+\epsilon_1 \cos\theta} . \quad -(42)$$

It is seen that the orbits are interlinked. For the N particle problem of the Newtonian limit:

$$\cos\theta = \frac{1}{\epsilon_1}\left(\frac{d_1}{R_1} - 1\right) = \dots = \frac{1}{\epsilon_i}\left(\frac{d_i}{R_i} - 1\right), \quad -(43)$$

$$i = 1, \dots, N,$$

so we reach the important conclusion that the N particle problem is soluble analytically, using computer algebra to deal with tedious complexity. For precessing orbits each with an x factor the constraint equation becomes:

$$\theta = \frac{1}{x_1}\cos^{-1}\left(\frac{1}{\epsilon_1}\left(\frac{d_1}{R_1} - 1\right)\right) = \dots = \frac{1}{x_i}\cos^{-1}\left(\frac{1}{\epsilon_i}\left(\frac{d_i}{R_i} - 1\right)\right),$$

$$i = 1, \dots, N. \quad -(44)$$

This method will produce an essentially infinite variety of previously unknown orbits.

Finally in this section the general analytical solution and procedure is illustrated with the four particle problem. The lagrangian is:

$$\mathcal{L} = \frac{1}{2}\left(m_1|\dot{\underline{r}}_1|^2 + m_2|\dot{\underline{r}}_2|^2 + m_3|\dot{\underline{r}}_3|^2 + m_4|\dot{\underline{r}}_4|^2\right) - \frac{m_1 m_2 G}{|\underline{r}_1 - \underline{r}_2|} - \frac{m_1 m_3 G}{|\underline{r}_1 - \underline{r}_3|} - \frac{m_2 m_3 G}{|\underline{r}_2 - \underline{r}_3|} - \frac{m_1 m_4 G}{|\underline{r}_1 - \underline{r}_4|} - \frac{m_2 m_4 G}{|\underline{r}_2 - \underline{r}_4|} - \frac{m_3 m_4 G}{|\underline{r}_3 - \underline{r}_4|} \quad -(45)$$

and is factorized into the sum of six two particle lagrangians:

$$\mathcal{L} = \frac{1}{3}\left(\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 + \mathcal{L}_6\right) \quad -(46)$$

where

$$\mathcal{L}_1 = \frac{1}{2}\left(m_1|\dot{\underline{r}}_1|^2 + m_2|\dot{\underline{r}}_2|^2\right) - 3\frac{m_1 m_2 G}{|\underline{r}_1 - \underline{r}_2|}$$

$$\mathcal{L}_2 = \frac{1}{2}\left(m_1|\dot{\underline{r}}_1|^2 + m_3|\dot{\underline{r}}_3|^2\right) - 3\frac{m_1 m_3 G}{|\underline{r}_1 - \underline{r}_3|}$$

$$\mathcal{L}_3 = \frac{1}{2}\left(m_2|\dot{\underline{r}}_2|^2 + m_3|\dot{\underline{r}}_3|^2\right) - 3\frac{m_2 m_3 G}{|\underline{r}_2 - \underline{r}_3|}$$

$$\mathcal{L}_4 = \frac{1}{2}\left(m_1|\dot{\underline{r}}_1|^2 + m_4|\dot{\underline{r}}_4|^2\right) - 3\frac{m_1 m_4 G}{|\underline{r}_1 - \underline{r}_4|}$$

$$\mathcal{L}_5 = \frac{1}{2}\left(m_2|\dot{\underline{r}}_2|^2 + m_4|\dot{\underline{r}}_4|^2\right) - 3\frac{m_2 m_4 G}{|\underline{r}_2 - \underline{r}_4|}$$

$$\mathcal{L}_6 = \frac{1}{2}\left(m_3|\dot{\underline{r}}_3|^2 + m_4|\dot{\underline{r}}_4|^2\right) - 3\frac{m_3 m_4 G}{|\underline{r}_3 - \underline{r}_4|} \quad -(47)$$

The six centres of mass of each pair of particles are defined by:

$$m_1 \underline{r}_1 + m_2 \underline{r}_2 = \underline{0}, \; R_1 = |\underline{r}_1 - \underline{r}_2|,$$
$$m_2 \underline{r}_2 + m_3 \underline{r}_3 = \underline{0}, \; R_2 = |\underline{r}_2 - \underline{r}_3|,$$
$$m_1 \underline{r}_1 + m_3 \underline{r}_3 = \underline{0}, \; R_3 = |\underline{r}_1 - \underline{r}_3|,$$
$$m_1 \underline{r}_1 + m_4 \underline{r}_4 = \underline{0}, \; R_4 = |\underline{r}_1 - \underline{r}_4|,$$
$$m_2 \underline{r}_2 + m_4 \underline{r}_4 = \underline{0}, \; R_5 = |\underline{r}_2 - \underline{r}_4|,$$
$$m_3 \underline{r}_3 + m_4 \underline{r}_4 = \underline{0}, \; R_6 = |\underline{r}_3 - \underline{r}_4|,$$

and the six reduced masses by:

$$\mu_1 = \frac{m_1 m_2}{m_1 + m_2}, \; \mu_2 = \frac{m_1 m_3}{m_1 + m_3}, \; \mu_3 = \frac{m_2 m_3}{m_2 + m_3},$$

$$\mu_4 = \frac{m_1 m_4}{m_1 + m_4}, \; \mu_5 = \frac{m_2 m_4}{m_2 + m_4}, \; \mu_6 = \frac{m_3 m_4}{m_3 + m_4} : \quad -(49)$$

Finally define six orbits:

$$R_i = \frac{\alpha_i}{1 + \epsilon_i \cos\theta}, \quad i = 1, .., 6 \quad -(50)$$

where:

$$\alpha_i = \frac{L_i^2}{\mu_i k_i}, \quad i = 1, \dots, 6 \quad -(51)$$

$$\epsilon_i = \left(1 + \frac{2 E_i L_i^2}{\mu_i k_i^2}\right)^{1/2}, \; i = 1, .., 6, \quad -(52)$$

in which the k constants are:

$$k_1 = 3 m_1 m_2 G, \; k_2 = 3 m_1 m_3 G, \; k_3 = 3 m_2 m_3 G,$$
$$k_4 = 3 m_1 m_4 G, \; k_5 = 3 m_2 m_4 G, \; k_6 = 3 m_3 m_4 G. \quad -(53)$$

The constraint equation is:

$$\cos\theta = \frac{1}{\epsilon_i}\left(\frac{\alpha_i}{R_i} - 1\right), \quad i = 1, \ldots, 6 \quad -(54)$$

giving:

$$R_{i+1} = \alpha_{i+1}\left(1 - \frac{\epsilon_{i+1}}{\epsilon_i}\left(\frac{\alpha_i - R_i}{R_i}\right)\right), \quad -(55)$$

$$i = 1, \ldots, 6 .$$

Therefore:

$$R_{i+2} = \alpha_{i+2}\left(1 - \frac{\epsilon_{i+2}}{\epsilon_{i+1}}\left(\frac{\alpha_{i+1} - R_{i+1}}{R_{i+1}}\right)\right), \quad -(56)$$

with R_{i+1} given by Eq. (55), and:

$$R_i = \frac{\alpha_i}{1 + \epsilon_i \cos\theta} . \quad -(57)$$

With computer algebra this procedure can be extended straightforwardly to N particles, thus providing the first analytical solution to the problem in nearly four hundred years. The only assumption is that the centres of mass of each pair can be defined in equations such as (30) to (32), and this can always be done.

3. GRAPHICAL ILLUSTRATIONS OF THE ANALYTICAL SOLUTION.

Section by Dr. Horst Eckardt.

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3 Graphical illustrations of the analytical solution

We first illustrate the solutions for non-precessing ellipses given by Eqs.(40-42). The interlinking gives three ellipses for the radial coordinates of centers of masses R_i as expected, see Fig. 1. For motion with precession, Eq.(44) has to be used, resolved for each R_i. This gives the well known precessing ellipses, Fig. 2, again for the centers of masses R_i. Please note that the coordinates R_i are not identical to the mass coordinates r_i.

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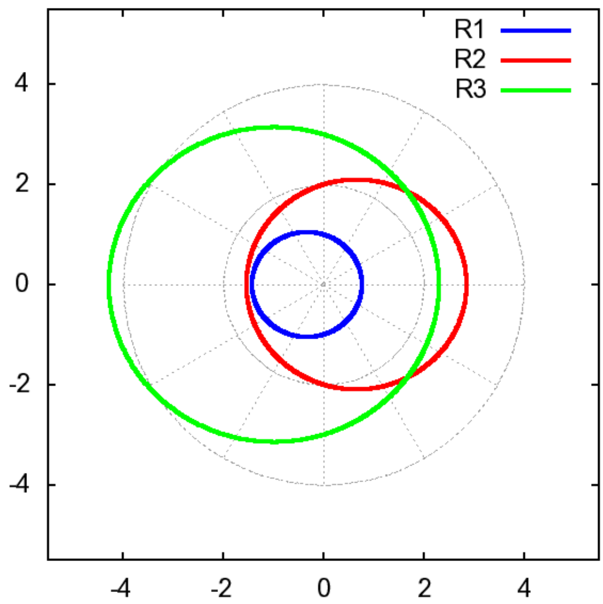

Figure 1: Ellipses $R_i(\theta)$ with parameters $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0.3, \alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3$.

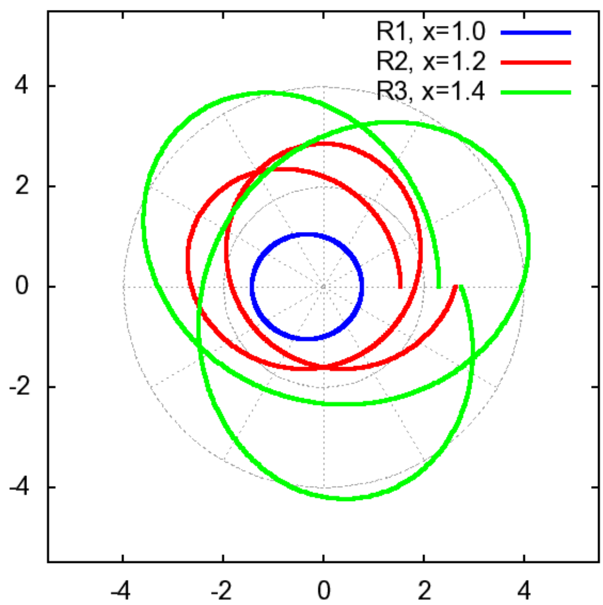

Figure 2: Precessing ellipses $R_i(\theta)$ with parameters same as for Fig. 1.

ACKNOWLEDGMENTS

The British Government is thanked for a Civil List pension and the staff of AIAS and many others for interesting discussions. Dave Burleigh is thanked for posting, and Alex Hill, Robert Cheshire and Simon Clifford for translation and broadcasting. The AIAS is established (2012) within the Newlands Family Trust.

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